On the Expressibility of Stochastic Switching Circuits

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Deterministic Switching Circuits

Foundation of modern digital circuit design: Any Boolean functions can be realized by deterministic switching relay circuits.

\[ z = x \lor y \]

[C. Shannon, 1938]
Stochastic Switching Circuits

What will happen if we replace deterministic switches with stochastic switches?

Examples

\[ P = \frac{1}{4} \]

\[ P = 1 - \frac{1}{2} \times \frac{1}{2} = \frac{3}{4} \]
Stochastic Switching Circuits

- Each switch (pswitch) is closed with some probability, chosen from pswitch set S.
- Probability $p$ can be realized if there is a stochastic switching circuit whose two terminals are connected with probability $p$. 
Stochastic Switching Circuits

- Similar to resistor circuits
  - Series:

    \[ p = p_1 p_2 \]

  ![Series Circuit Diagram]

- Parallel:

    \[ p = 1 - (1 - p_1)(1 - p_2) = p_1 + p_2 - p_1 p_2 \]

  ![Parallel Circuit Diagram]
sp circuits and ssp circuits

- **Series-Parallel (sp) circuit**

  \[
  sp = \begin{cases} 
  \text{sp series } sp \\
  \text{sp parallel } sp \\
  \text{pswitch}
  \end{cases}
  \]

- **Simple Series-Parallel (ssp) circuit**

  \[
  ssp = \begin{cases} 
  \text{ssp series } \text{pswitch} \\
  \text{ssp parallel } \text{pswitch} \\
  \text{pswitch}
  \end{cases}
  \]
sp circuits and ssp circuits

- **Series-Parallel (sp) circuit**

\[ sp = \begin{cases} 
    sp \text{ series } sp \\
    sp \text{ parallel } sp \\
    ps\text{witch} 
\end{cases} \]

- **Simple Series-Parallel (ssp) circuit**

\[ ssp = \begin{cases} 
    ssp \text{ series } ps\text{witch} \\
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    ps\text{witch} 
\end{cases} \]
Given pswitch set $S = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \right\}$, only probabilities $\frac{x}{q^n}$ can be realized.

1. Can all $\frac{a}{q^n}$ with $0 < a < q^n$ can be realized by an ssp circuit?
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2. How many pswitches are sufficient?
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1. Can all $\frac{a}{q^n}$ with $0 < a < q^n$ can be realized by an ssp circuit?
2. How many pswitches are sufficient?
3. How to approximate probabilities?
Wilhelm and Bruck’s Work


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<thead>
<tr>
<th>(q)</th>
<th>all (\frac{a}{q^n}) can be realized?</th>
<th>upper bound of circuit size</th>
</tr>
</thead>
<tbody>
<tr>
<td>2</td>
<td>yes, ssp circuit</td>
<td>(n)</td>
</tr>
<tr>
<td>3</td>
<td>yes, ssp circuit</td>
<td>(n)</td>
</tr>
<tr>
<td>4</td>
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<td>(2n - 1)</td>
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Wilhelm and Bruck’s Work


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<tr>
<td>$q = 5$</td>
<td>?</td>
<td></td>
</tr>
<tr>
<td>$q = 6$</td>
<td>?</td>
<td></td>
</tr>
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We generalized Wilhelm and Bruck’s work for \( q = 2 \) and \( q = 4 \):

\[
S = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \right\}
\]

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We want to realize $p_0 = \frac{a}{q^n}$:

- Define characteristic function $d\left(\frac{a}{q^n}\right) = \frac{q^{n-1}}{\gcd(a, q^{n-1})}$.
- We can add the last pswitch in the following way, where $2^s \leq q < 2^{s+1}$.

- Repeat the process above until $d(p_m) = 1$. 

**q is Even**

$\frac{1}{2}$ $\frac{2^s}{q}$

$\frac{1}{2}$ $\frac{q-2^s}{q}$
**q is Even**

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  - $p_1 < \frac{1}{2}$, $d(p_1)$ is even

    ![Diagram](image1)

  - $p_1 > \frac{1}{2}$, $d(p_1)$ is even

    ![Diagram](image2)

  - $p_1 < \frac{1}{2}$, $d(p_1)$ is odd

    ![Diagram](image3)

  - $p_1 > \frac{1}{2}$, $d(p_1)$ is odd

    ![Diagram](image4)

- Repeat the process above until $d(p_m) = 1$. 


Suppose the pswitch set is \( \{ \frac{1}{10}, \frac{2}{10}, \ldots, \frac{9}{10} \} \) and the desired probability is \( \frac{71}{100} \).
$q$ is Even, Example

Realization of $\frac{71}{100}$ when $q = 10$

$\frac{71}{100}, d = 10$
$q$ is Even, Example

Realization of $\frac{71}{100}$ when $q = 10$

$\frac{21}{50}$, $d = 5$
$q$ is Even, Example

Realization of $\frac{71}{100}$ when $q = 10$

$\frac{21}{40}, d = 4$
$q$ is Even, Example

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$\frac{1}{20}, d = 2$
$q$ is Even, Example

Realization of $\frac{71}{100}$ when $q = 10$
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Why all $\frac{a}{q^n}$ can be realized when $q$ is even?

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$d(p) \geq 1$ is an integer.
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Why all \( \frac{a}{q^n} \) can be realized when \( q \) is even?

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- At the beginning, \( d(p_0) \leq q^{n-1} \).
- In each step, if \( d(p_k) > 1 \), we have \( d(p_{k+1}) < d(p_k) \).
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- $\exists m$ s.t. $d(p_m) = 1$, where $p_m$ can be realized with single pswitch.
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- $\exists m$ s.t. $d(p_m) = 1$, where $p_m$ can be realized with single pswitch.
- Required $m + 1$ pswitches, carefully calculate the number of steps we have

$$m \leq \lceil \log_2 q \rceil (n - 1)$$
Similarly, we can also get

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### $q \geq 3$ is a Prime Number

- But not every $q$ can work:

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For example, given \( S = \{ \frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}, \frac{7}{5^2} \} \) cannot be realized by an sp circuit.
### Summary of Current Results

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For some desired probabilities, they can never be realized using pswitch set $S = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \right\}$. In this case, it is necessary to construct a circuit to get a good approximation of the desired probability.
Circuits for Approximating Probabilities

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- \( p_a \) : the approximation probability that can be realized.
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- $p_d$: the desired probability
- $p_a$: the approximation probability that can be realized.

**Theorem**

Given a pswitch set $S = \{ \frac{1}{q}, \frac{2}{q}, ..., \frac{q-1}{q} \}$, for any desired probability $p_d$, there exists a rational probability $p_a$ such that $|p_a - p_d| \leq \frac{1}{2q^n}$ and $p_a$ can be realized by an ssp circuit with at most $2n - 1$ pswitches.
Circuits for Approximating Probabilities

- $2n - 1$ pswitches:

\[
0 \quad \Delta^{(2n-1)} \quad \ldots \quad 1
\]

- probability can be realized with at most $2n - 1$ pswitches.

- $\Delta^{(2n-1)}$: the maximal difference between two neighbor probabilities.

Assume the statement is true, then

\[
\Delta^{(2n-1)} \leq \frac{1}{q^n}
\]
Circuits for Approximating Probabilities

- Add 2 more pswitches, where $u \in \{0, 1, \ldots, q - 1\}$

- New probabilities (linear mapping)
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\[ p_{2n+1} \]

Red Intervals: $\Delta^{(2n+1)} \leq \Delta^{(2n-1)} \times \frac{1}{q}$
Circuits for Approximating Probabilities

- Add 2 more pswitches, where \( u \in \{0, 1, \ldots, q - 1\} \)

\[
p_{2n-1} \quad \frac{u+1}{q} \quad \frac{q-1}{q} \quad p_{2n+1}
\]

- New probabilities (linear mapping)

\[
\begin{align*}
0 & \quad \ldots & \quad 1 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
& \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
& \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
& \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
& \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
& \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet
\end{align*}
\]

\[
\begin{align*}
0 & \quad \frac{1}{q} & \quad \frac{2}{q} & \quad \ldots & \quad 1 \\
\bullet & \quad \bullet & \quad \bullet & \quad \bullet & \quad \bullet \\
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\[
A \xrightarrow{p_{2n-1}} u+1 \xrightarrow{q^{-1}} B \rightarrow p_{2n+1}
\]

- New probabilities (linear mapping)
Circuits for Approximating Probabilities

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Circuits for Approximating Probabilities

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Green Intervals: \( \Delta^{(2n+1)} \leq \Delta^{(2n-1)} \times \frac{1}{q} \)
Circuits for Approximating Probabilities

- New probabilities (linear mapping)

\[ p_2^n - 1 \leq \ldots \leq p_2^n + 1 \]

\[ (0, 1) \text{ is covered by the red and green intervals.} \]

\[ \Delta^{(2n+1)} \leq \Delta^{(2n-1)} \times \frac{1}{q} \leq \frac{1}{q^{n+1}}. \]

\[ |p_d - p_a| \leq \Delta^{(2n+1)}/2 \leq \frac{1}{2q^{n+1}} \]
Approximate $p_{2n+1} = \frac{a}{q^n}$

- If $p_{2n+1} \in \left[ \frac{u}{q}, \frac{u}{q} + \frac{1}{q} - \frac{u}{q^2} \right]$ for some $u \in \{0, 1, \ldots, q - 1\}$, add the last two pswitches:
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  ![](diagram1.png)

- If $p_{2n+1} \in \left[ \frac{u}{q} + \frac{1}{q} - \frac{u}{q^2}, \frac{u+1}{q} \right]$ for some $u \in \{0, 1, ..., q - 1\}$, add the last two pswitches:

  ![](diagram2.png)
Approximate $p_{2n+1} = \frac{a}{q^n}$

- If $p_{2n+1} \in \left[\frac{u}{q}, \frac{u}{q} + \frac{1}{q} - \frac{u}{q^2}\right]$ for some $u \in \{0, 1, \ldots, q - 1\}$, add the last two pswitches:

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- Repeat this process for $(n - 1)$ times, we can get $p_{2n-1}, p_{2n-3}, \ldots, p_1$. 
Approximate $p_{2n+1} = \frac{a}{q^n}$

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  ![Diagram 1]

- If $p_{2n+1} \in \left[ \frac{u}{q} + \frac{1}{q} - \frac{u}{q^2}, \frac{u+1}{q} \right]$ for some $u \in \{0, 1, \ldots, q - 1\}$, add the last two pswitches:

  ![Diagram 2]

- Repeat this process for $(n - 1)$ times, we can get $p_{2n-1}, p_{2n-3}, \ldots, p_1$.
- Replace $p_1$ with nearest single pswitch.
Example of Approximating Probabilities

\[ p_d^{(5)} = \frac{3}{7}, \quad S = \{\frac{1}{5}, \frac{2}{5}, \frac{3}{5}, \frac{4}{5}\} \]

\[ p_a^{(5)} = 0.42784, \quad |p_d^{(5)} - p_a^{(5)}| = 7.3 \times 10^{-4} \]
Summary

- Given pswitch set $S = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \right\}$
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  - If \( q > 3 \) is a prime number, \( \exists \frac{a}{q^n} \) that cannot be realized by an sp circuit (ssp circuit).
  - Other cases, such as \( q = 35 \), open question.
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- Other cases, such as $q = 35$, open question.
- $\forall q$, any desired probability can be approximated by $2n - 1$ pswitches with error $\leq \frac{1}{2q^n}$. 
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  - $\forall q$, any desired probability can be approximated by $2^n - 1$ pswitches with error $\leq \frac{1}{2q^n}$.

- Next talk, by Po-Ling Loh
Summary

- Given pswitch set $S = \left\{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \right\}$
  - If $q \mod 2 = 0$ or $q \mod 3 = 0$, all rational $\frac{a}{q^n}$ can be realized by an ssp circuit.
  - If $q > 3$ is a prime number, $\exists \frac{a}{q^n}$ that cannot be realized by an sp circuit (ssp circuit).
  - Other cases, such as $q = 35$, open question.
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- Next talk, by Po-Ling Loh
  - Applications of stochastic switching circuits
Given pswitch set $S = \{ \frac{1}{q}, \frac{2}{q}, \ldots, \frac{q-1}{q} \}$

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Next talk, by Po-Ling Loh

- Applications of stochastic switching circuits
- Robustness of stochastic switching circuits
Thank You!!!