Homework # 1
Due Tuesday April 15, 2014, at 2:30 PM
Collaboration and discussions are not allowed on Problem 1
and are allowed and encouraged on Problem 2

1. Playing with Syntax Boxes (Collaboration is not allowed on this Problem)
   In this problem we learn how to compute with syntax boxes, that we call s-boxes. An s-box has multiple inputs and a single output (denoted by $o$), where the output is a function of the inputs. For example, the following table represents an s-box with two inputs:

   $s_1(a, b) = \begin{array}{c|c|c|c}
   a & b & o \\
   \hline
   0 & 0 & 0 \\
   0 & 1 & 1 \\
   1 & 0 & 1 \\
   1 & 1 & 1 \\
   \end{array}$

   (a) By composing s-boxes we can get new s-boxes, for example, let $s_2(a, b, c) = s_1(s_1(a, b), c)$. Write the table for $s_2(a, b, c)$.

   (b) Using any number of $s_1$ boxes, can you compute the following $s_3(a, b)$? Justify your answer.

   $s_3(a, b) = \begin{array}{c|c|c|c}
   a & b & o \\
   \hline
   0 & 0 & 1 \\
   0 & 1 & 0 \\
   1 & 0 & 0 \\
   1 & 1 & 1 \\
   \end{array}$

   (c) Let $s_4(a, b)$ be the following s-box,

   $s_4(a, b) = \begin{array}{c|c|c|c}
   a & b & o \\
   \hline
   0 & 0 & 1 \\
   0 & 1 & 1 \\
   1 & 0 & 1 \\
   1 & 1 & 0 \\
   \end{array}$

   Using any number of $s_4$ boxes, can you compute $s_1(a, b)$? Justify your answer.

   (d) Using any number of $s_4$ boxes, can you compute $s_3(a, b)$? Justify your answer.
2. **Rolling Dice** *(Collaboration is allowed on this Problem)*

As you know, dice are little cubes whose faces are numbered from one (denoted as \([1]\)) to six (denoted as \([6]\)) using small dots. We represent the conventional die, called \(A\) by:

\[
A = \{[1], [2], [3], [4], [5], [6]\}
\]

Rolling dice provided random numbers for gambling and games as early as 5,000 years ago. Are you ready to play?

(a) You are rolling two *fair* dice like \(A\). Note that in a *fair* die each side has equal probability. Consider the sum of the two dice - there are 11 possible sums, \([2]\) to \([12]\).

i. What is the most likely sum? What is the probability that you get the most likely sum?

ii. What is the expected value of the sum?

Justify your answers.

(b) Consider the following two new kinds of dice.

\[
B = \{[2], [2], [2], [2], [5], [5]\}
\]

\[
C = \{[1], [1], [4], [4], [4], [4]\}
\]

Both \(B\) and \(C\) are *fair* dice. Notice that \(B\) has four sides that are \([2]\) and two sides that are \([5]\), and \(C\) has two sides that are \([1]\) and four sides that are \([4]\).

You roll the dice \(B\) and \(C\):

i. What is the probability that \(B > C\)？

ii. What is the probability that the sum of \(B\) and \(C\) is \([6]\)? What is the expected value of the sum?

Justify your answers.

(c) In the last Super Bowl, the referee could not find a coin for the coin toss; the umpire offered him a die.

i. Assuming that the die is \(A\) and it is fair, suggest a method to use the die for the coin toss.

ii. Assuming that the die is \(A\) and it is biased (the probabilities of the different sides are not equal), however, you do not know the bias; suggest a method to use the die for the coin toss. *Hint:* Your method should be based on multiple rolls of the die.

iii. Assuming that the die is \(B\) and it is fair, suggest an optimal method (in terms of the expected number of rolls) to use the die for a coin toss. Compute the expected number of rolls to generate a coin toss. If you use the analogous method from (c)ii (i.e. not optimized for die \(B\)), what is the expected number of rolls?

Justify your answers.