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Boole's Algebra Isn't Boolean Algebra

*A description, using modern algebra,
of what Boole really did create.*

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To Boole and his mid-nineteenth century contemporaries, the title of this article would have been very puzzling. For Boole's first work in logic, *The Mathematical Analysis of Logic*, appeared in 1847 and, although the beginnings of modern abstract algebra can be traced back to the early part of the nineteenth century, the subject had not fully emerged until towards the end of the century. Only then could one clearly distinguish and compare algebras. (We use the term **algebra** here as standing for a formal system, not a structure which realizes, or is a model for, it—for instance, the algebra of integral domains as codified by a set of axioms *versus* a particular structure, e.g., the integers, which satisfies these axioms.) Granted, however, that this later full degree of understanding has been attained, and that one can conceptually distinguish algebras, is it not true that Boole's "algebra of logic" is Boolean algebra?

Conventional opinion without exception is on the affirmative side of this question. Briefly put, this opinion has it that Boole was the first to give a successful algebraic setting for doing logic (of class terms, or properties) and, as is well known, the abstract form of the calculus of classes (classes = sets), is Boolean algebra. But if we look carefully at what Boole actually did—his major work is *The Laws of Thought* of 1854—we find him carrying out operations, procedures, and processes of an algebraic character, often inadequately justified by present-day standards and, at times, making no sense to a modern mathematician. Not only that, but even though his starting algebraic equations (expressing premises of an argument) were meaningful when interpreted in logic, the allowed transformations often led to equations without meaning in logic. Boole considered this acceptable so long as the end result could be given a meaning, which he did give by a specifically introduced procedure. Writing in 1870 the logician (and economist) W. S. Jevons complained:

...he (Boole) shrouded the simplest logical processes in the mysterious operations of a mathematical calculus. The intricate trains of symbolic transformations, by which many of the examples in the *Laws of Thought* are solved, can be followed only by highly accomplished mathematical minds; and even a mathematician would fail to find any demonstrative force in a calculus which fearlessly employs unmeaning and incomprehensible symbols, and attributes a signification to them by a subsequent process of interpretation [4, p. 143].

As this quote indicates, it appears that to determine what sort of algebra Boole did use to do logic will require a fairly substantial exegetical effort, and what the outcome will be, whether Boolean algebra or something else, is not immediately clear. Before entering into a presentation of Boole's ideas, we depict the foundations-of-algebra *milieu* in which he worked.

Symbolical Algebra (circa 1830-1840)

Although in 1837 W. R. Hamilton had shown how to define complex numbers as ordered pairs of ordinary numbers, there was, as yet, no real understanding of the various number systems as being successive, more inclusive, extensions of natural numbers; nor was there any conception of such number structures as models of appropriate formal algebraic systems. Such fundamental algebraic properties as commutativity and distributivity had only recently been so named (by F-J.



GEORGE BOOLE, F.R.S.

Engraving reproduced from "George Boole, F. R. S. 1815-1864," by G. Taylor, *Notes and Records of the Royal Society of London*, (12) 1956, 44-52.

Servois in 1814/15), and associativity not until 1844 (by Hamilton). And, despite the fact that over the course of several centuries the techniques and application of numerical algebra had become highly developed and refined, there was no adequate body of justifying principles. Symbolical Algebra, as it was called, claimed to provide these absent principles. Its chief promulgator was George Peacock, Fellow and Tutor of Trinity College, Cambridge. Also prominent in its advocacy were Duncan F. Gregory, editor of the journal which published Boole's earliest mathematical research, and Augustus De Morgan, an early champion of Boole's work in logic, who also wrote extensively on formal logic.

The doctrine espoused by Peacock was that there were two sciences of algebra, *arithmetical* and *symbolical*. In the former the general symbols and operations refer to the numbers and operations of "common arithmetic" whose meaning required in many cases restriction on the performability of the operations. For example, in arithmetical algebra one couldn't subtract a larger from a smaller number and hence the form $a - b$ involving the general symbols a and b could be meaningless if so interpreted. In symbolical algebra, however, all restrictions on operations are removed—what meaning expressions then had was to be subsequently determined from the assumed laws of symbols. The laws are obtained from the following principles (stated in Peacock's language to retain historicity):

(i) Whatever forms in general symbols are equivalent in arithmetical algebra, are also equivalent in symbolical algebra.

(ii) Whatever forms are equivalent in arithmetical algebra when the symbols are general in their form, though specific in their value, will continue to be equivalent when the symbols are general in their nature as well as in their form.

Thus although $a(b - c) = ab - ac$ is true in arithmetical algebra only if b is not less than c , this restriction is removed in symbolical algebra and the equation is considered to be true. Symbolical algebra is to be an extension of the arithmetical in the sense that when its symbols are numbers and its operations arithmetical, the results must be identical with that of arithmetic. However, "...inasmuch as in many cases, the operations required to be performed are impossible, and their results inexplicable, in their ordinary sense, it follows that the meaning of the operations performed, as well as the results obtained under such circumstance, must be derived from the assumed rules, and not from their definitions or assumed meanings, as in Arithmetical Algebra." [9, p. 7] In contrast to this view, contemporary mathematics does not consider formal rules (or axioms) as being able to create meanings for operations, only to set limits for them—e.g., adoption of $a + b = b + a$ excludes the symbol $+$ from being interpreted as a noncommutative operation. It should be emphasized that nowhere in Peacock's *Treatise* do we find a full listing of algebraic properties of the operations analogous to a present day set of axioms.

Despite such inadequacy, and much more that one could find fault with, symbolical algebra did provide an environment in which algebra could be freed from its exclusive use with number. For example, it provided a measure of justification for the calculus of operations or separation of symbols, as it was also known. In this calculus one separates off the symbols of operation and performs algebraic operations on them. A simple example is the well-known symbolic formula $(D + a)^2 = D^2 + 2aD + a^2$ of elementary differential equations in which D stands for the operation of differentiation and the square on D indicates a two-fold application. This type of procedure, of applying algebra to "symbols" and using special interpretation, was suspect in the view of some mathematicians—Cauchy for one—but was widely used in England. Boole was a leading practitioner of the art.

Though a confirmed adherent of the symbolical algebra school of thought, Boole was no rigid follower. He significantly modified Peacock's principles by relaxing the requirement that when the general symbols and the operations have their common arithmetical meaning, the result should be an arithmetical truth. For example, in a paper of 1844 for which he received a Royal Society medal, Boole introduced noncommuting types of operators. Turning to the immediate subject at hand, the characteristic law $x^2 = x$ which he added to algebra so as to do logic is not true arithmetically save—as Boole points out—if one restricts x to being 0 or 1. Did Boole thus create

Boolean algebra by adding $x^2 = x$? Before addressing ourselves to the question we turn to a brief exposition of Boolean algebra.

Boolean Algebra

Nowadays we describe a formal algebraic system by way of axioms. For Boolean algebra especially, such axiom systems come in a large variety of shapes (basic operations, relations) and sizes (number of axioms). The one we choose to present here focuses attention on analogies with numerical algebra. It uses two binary operation symbols, $+$ and $\times$, one unary operation symbol $'$, and two constant symbols 0 and 1.

Axioms for Boolean Algebra

- | | |
|----------------------------|-------------------------------|
| (i) $ab = ba$, | $a + b = b + a$, |
| (ii) $(ab)c = a(bc)$, | $(a + b) + c = a + (b + c)$, |
| (iii) $1a = a$, | $0 + a = a$, |
| (iv) $aa' = 0$, | $a + a' = 1$, |
| (v) $a(b + c) = ab + ac$, | $a + (bc) = (a + b)(a + c)$, |
| (vi) $aa = a$, | $a + a = a$. |

The evident analogy with ordinary algebra which is displayed here can be further enhanced by the introduction of a subtraction symbol in terms of which $'$ can be expressed: we let $x - y$ stand for xy' so that, with (iii), we have $1 - x = 1x' = x'$. Hence in place of (iv) we can write

$$(iv^*) \quad a(1 - a) = 0, \quad a + (1 - a) = 1.$$

We need to distinguish a **Boolean algebra** from the *general concept of*, or the *formal theory of*, Boolean algebras. When using the indefinite article, we are referring to a particular mathematical structure which, via an appropriate interpretation, satisfies the above axioms. The best known example of a Boolean algebra is the set of subsets of a fixed set, with the set operations of union, intersection and complementation interpreting, respectively, the symbols $+$, $\times$, $'$, and with the empty set and the fixed set interpreting 0 and 1.

One can come to Boolean algebra from another direction which adds a little more to the analogy with numerical algebra. The Boolean addition in the above axioms lacks the property of linear solvability or, equivalently, existence of an additive inverse; that is, $a + x = b$ need not have a solution for x nor, if it did, a unique one. If, however, one introduces by definition an operation $+_{\Delta}$ ("symmetric difference") by putting

$$x +_{\Delta} y = x - y + y - x = xy' + yx',$$

then it is true that in any Boolean algebra the equation $a +_{\Delta} x = b$ has a unique solution for x , for any a and b . One can show that, under the operations $+_{\Delta}$ and $\times$, with 0 and 1 as the zero and unit, a Boolean algebra is a commutative ring with unit. A commutative ring having the idempotency property $a^2 = a$ is called a **Boolean ring**. It is straightforward to show that the theories of Boolean algebra and Boolean rings (with unit) are equivalent. The particular axiom system given above for Boolean algebras has the property that it becomes an axiom system for Boolean rings just by replacing the axiom $a + a = a$ by $a + a = 0$ (with all the $+$'s then taken to be symmetric difference). A well-known simple example of a mathematical structure which is a Boolean ring is the field of integers modulo 2.

The close affinity of any Boolean algebra with two-valued arithmetic is expressed, in sophisticated language, by the following representation theorem [10, p. 50]: *every (non-degenerate) Boolean algebra is isomorphic to a subalgebra of a direct union of two-element Boolean algebras*. In unsophisticated language this says that any Boolean algebra "looks like" an algebra (Boolean, of course) on a subset of all n -tuples $U = \{\langle v_1, v_2, \dots \rangle\}$ with each v_i either 0, or 1, (i the zero and unit of a two-element Boolean algebra), and with operations on the n -tuples defined componentwise. The importance of the two-valued arithmetic in connection with Boolean algebras is also brought out in the following useful result: any Boolean polynomial equation in n variables is true

for all Boolean algebras (i.e., is a Boolean identity) if true for all 2^n possible assignments of 0's and 1's for the variables. We also note that by virtue of idempotency ($x^2 = x$) a Boolean polynomial is linear in each of its variables.

Since a Boolean algebra can be viewed as a ring (with symmetric difference as the addition) the notions of ideals and of residue classes can be introduced in customary fashion.

Boole's Logic of Class Terms

Boole believed he was the first to use a mathematical approach to logic. But much earlier Leibniz had conceived of the idea of a formal mathematical system which could be used to conduct logical inference. In manuscript notes and papers—which did not become generally known until some 200 years later—he had made some remarkable starts but did not bring the project to fruition. One of the minor hampering items may have been his regarding general terms (e.g., “human,” “sheep”) primarily as designating attributes rather than classes (i.e., thinking *intensionally* rather than *extensionally*). In this view the compound term “human sheep” has a wider, more inclusive, intension than either “human” or “sheep,” whereas as a class it has a less inclusive extension than either of the component terms. Thinking extensionally is far simpler; for example, the terms “human” and “featherless biped” designate distinct attributes but as classes are identical (plucked chickens being ruled out). Leibniz could, and did, think either way, but Boole's thinking was exclusively extensional.

A more substantial difference, however, was the circumstance that Leibniz, faced with *infiniti modi calculandi*, had to grope his way towards a calculus of logic, whereas Boole was sure he had one ready-made, namely symbolical algebra, furnished with a suitable interpretation for its general symbols and operations. Having as a guide a theoretical system, even if only approximately correct, can be decidedly advantageous. As we now know, neither of them had at their disposal a clear and sufficiently extensive body of formal logical usage—what was available, to Boole as well as to Leibniz, wasn't much more than the Aristotelian syllogistic, a body of rules not lending itself well to formulation as an algebra of the equational type, and, moreover, burdened with the distracting problem of existential import (Does “All S are P ” imply “There are S ?”).

In his initial writings on logic Boole associated the general symbols $x, y, z, \dots$ of symbolical algebra with *operators* which selected classes out of a universal class. (Subsequently, this aspect became less pronounced and the symbols came to stand for the classes themselves.) Selecting out the x 's and then the y 's produces the same class as selecting out the y 's and then the x 's. Thus Boole wrote

$$xy = yx.$$

For Boole to think of using multiplication for logical intersection was quite natural since this was what was used in the calculus of operators to indicate successive application of operations. In keeping with this “selecting out” idea the symbolic laws

$$x \cdot 1 = x, \quad x \cdot 0 = 0$$

inevitably suggest the logical interpretation for 1 as the universal class and 0 as the empty class. Selecting out the x 's and then from this the x 's yields the x 's; hence the algebraic property,

$$xx = x^2 = x,$$

a departure from ordinary algebra which we have already mentioned.

To symbolize the aggregate of two classes (e.g., “mountains and vales”) Boole used the symbol of addition but with the proviso that the classes be disjoint and hence the operation is only partially defined. Boole had no objection to partially defined operations—compare, for instance, $a - b$ in arithmetical algebra. Similarly, subtraction required that the “subtrahend” class be contained in the “diminunend” class. Under these conditions he found

$$x(y + z) = xy + xz$$

and

$$x(y - z) = xy - xz$$

are true for classes. With hardly much more than these examples Boole boldly proposed [1, p. 37]:

Let us conceive, then, of an Algebra in which the symbols, x, y, z , etc., admit indifferently of the values 0 and 1, and of these values alone. The laws, the axioms, and the processes, of such an Algebra will be identical in their whole extent with the laws, the axioms, and the processes of an Algebra of Logic. Difference of interpretation will alone divide them. Upon this principle the method of the following work is established.

In these sentences we have, for the period, some remarkably innovative ideas: that there can be an algebra of 0 and 1, that such an algebra has at least two interpretations, and that one of these is logic. These items give rise to the following questions.

- (1) What is Boole's algebra of 0 and 1?
- (2) Is arithmetic modulo 2 one of the interpretations?
- (3) How does one interpret the algebra to get logic?

It will be convenient for us to address these questions in reverse order.

Re (3). As already indicated, the symbols x, y, z , etc., are to stand for classes, with 1 being the universe and 0 the empty class. The product xy represents the intersection of the classes represented by x and y and is meaningful in all circumstances. The aggregate class $x + y$ has meaning only if x and y are mutually exclusive; likewise $x - y$ only if y is contained in x (and thus is *not* the same as the xy' of Boolean algebra). The expression $1 - x$ is meaningful in any case and represents the complement of x in the universe 1. Assertions are always represented by equations—e.g., “No x 's are y 's” is “ $xy = 0$ ” and “All x 's are y 's” is “ $x = vy$ ” where v is an indefinite class symbol. This use of the special symbol v was adversely criticized by other logicians but can in part be justified by considering it as an understood existentially quantified variable [2, pp. 97–98]. Logical inference—about which we will say very little in this paper—is performed by algebraic transformations on equations, allowing all the operations of “common” algebra. This creates a problem (to be discussed in detail in our next section) in that expressions arise, e.g., $-1, x + x, x - 1, x/y$, to which no logical meaning has been assigned. Boole considered this perfectly acceptable in intermediate steps of a deduction so long as the starting and ending equations were interpretable in logic. He likened this to “the employment of the uninterpretable symbol $\sqrt{-1}$ in the intermediate processes of trigonometry.” Aside from this dubious feature, the direct translating of logical statements involving class terms *into* algebraic form goes tolerably well.

Re (2). It is well known that an interpretation for Boolean algebra (i.e., the algebra of section 2) can be given as an algebra over the two-element set $\{0, 1\}$ whose arithmetic differs from ordinary arithmetic in that one has either (i) $1 + 1 = 1$ or (ii) $1 + 1 = 0$, the latter case yielding arithmetic modulo 2. However, neither (i) nor (ii) can be the interpretation Boole has in mind, for he always writes “ $1 + 1$ ” as “2”, considering 2 as a “numerical” factor which does not obey the general law of thought $x^2 = x$. Correspondingly, for any algebraic expression A , the sum “ $A + A$ ” is written “ $2A$ ” and never simplified. In a letter to Jevons written in 1863 in response to an inquiry, Boole flatly asserted that $x + x = x$ was not true (i.e., a law) in logic, so that (by implication) $1 + 1 = 1$ would not be true in his algebra of 0 and 1. Clearly then Boole would consider neither arithmetic modulo 2 nor the alternative with $1 + 1 = 1$ as an interpretation for his algebra.

Re (1). It is not easy to determine what Boole's algebra is since he gives no explicit list of its laws (i.e., axioms). He talks about $x^2 = x$ being the general law of thought, but $A^2 = A$ for arbitrary A is not one of his algebraic laws—it holds for *class symbols* x, y , etc., and for expressions such as

$$xy, 1 - x, xy' + x'y$$

which he refers to as “independently interpretable,” but not for expressions such as

$$2x, x + y, x - y$$

which nevertheless do occur in his procedures. An expression A not satisfying the idempotency condition $A^2 = A$ is dubbed “uninterpretable.” A careful examination of those algebraic properties

of $+$ and $\times$ which Boole actually makes use of, for interpretables as well as uninterpretables, show that they are those of a

commutative ring with unit having no additive or multiplicative nilpotents. (SM)

(No additive or multiplicative nilpotents means that $nA = 0 \Rightarrow A = 0$ and that $A^n = 0 \Rightarrow A = 0$.) To use this algebra for logic (of class terms) one needs entities which satisfy Boole's law of thought $x^2 = x$. Let us single out and call such entities the Boolean (or idempotent) elements and designate the set of such elements by $\mathbf{B}$. Clearly the 0 and 1 of the ring are in $\mathbf{B}$. Also, for any $x, y \in \mathbf{B}$, the commutative ring properties imply that the elements

$$xy, 1 - x, xy' + x'y, \text{ and } xy + x'y + xy'$$

(here $x' = 1 - x$) are also in $\mathbf{B}$. It is easy to show that the elements of $\mathbf{B}$ will constitute a Boolean algebra (or equivalently a Boolean ring) with the operation

$$x \cup y = xy + x'y + xy'$$

as the Boolean sum, or with

$$x +_{\Delta} y = x'y + xy'$$

as the Boolean ring addition. Thus by restricting himself to the idempotents of his algebra, Boole would have had Boolean algebra—and we rightfully honor him by attaching his name to this algebra. Only Boole didn't know it. He steadfastly refused to acknowledge any operation but his $+$. When Jevons claimed that $x + x = x$, and Boole emphatically denied this, they were really talking about different operations. Jevons' $+$ is indeed the present day $\cup$; however Boole's $+$ is not, as is generally believed, the $+_{\Delta}$ of the Boolean ring but the $+$ of the ring SM. This algebra merits a brief discussion.

What kinds of models (realizations) does Boole's algebra SM have? An answer to this question comes from a structure theorem of N. H. McCoy [8, p. 123] which, adapted to our circumstance reads: any model of SM is isomorphic to a subdirect sum of integral domains which are without additive nilpotents. Intuitively such a model looks like n -tuples of elements in which each component ranges over an integral domain without nilpotents, and with the operations of $+$ and $\times$ for these n -tuples defined componentwise.

Not all models or interpretations of a set of axioms are of equal general interest. In the case of Boolean algebra there is a sense in which one can say that the principal interpretation is that of an algebra of sets. For Boole's algebra SM we consider the principal interpretation to be that of an **algebra of signed multisets**: a multiset is like an ordinary set except that multiple occurrences of elements are allowed, and by a signed multiset we mean one in which negative multiplicities are allowed. For example, during a poker game your pile of chips contains various positive or zero multiples of red, white or blue chips, all of each color indistinguishable as far as the game is concerned; and if you borrow from the pot, you are adding negative multiples to your pile. When the multiplicities are restricted to being either 0 or 1, then multisets become ordinary sets. As for the operations, Boole's $+$ corresponds to dumping the contents of the two multisets together, and $\times$ corresponds to multiplying the respective multiplicities. (For more details see [2, pp. 91, 92–96]; in that work we used the term “heap,” not realizing that “multiset” was already in use.)

If all one wanted to use Boole's algebra for was to do class (i.e., term) logic, then there is no need to go beyond the idempotents—and this is the path history chose. Nevertheless, the richer structure of Boole's algebra can be of interest in its own right, for example, in the logic of multisets. As a simple instance, the equivalence

$$(x + x)y = 0 \Leftrightarrow xy = 0$$

when interpreted for multisets tells us that duplicating the elements of a set does not affect its being exclusive from another set. It is possible that Boole's algebra can be useful in the subject of pseudo-Boolean functions [3]. Pseudo-Boolean functions, a notion used in operations research, are functions from a two-element set $\{0, 1\}$ into the integers, and they have an obvious representation by means of a polynomial in Boolean variables with integer coefficients.

Boole's Interpretation Procedure

As we have mentioned earlier, Boole considered his algebraic methods for doing logic to be sound so long as the starting and end formulas were interpretable in logic. But many of the algebraic processes—especially division, which plays an important role in Boole's method—led to uninterpretable forms. For example, in order to determine what the Boolean equation $xw = y$ implies about the class w , Boole solves this equation for w , obtaining

$$w = \frac{y}{x}.$$

What does y/x mean in logic? What is the interpretation for this equation? Boole ignores the first of these questions, but to handle the second he introduces the idea of the **expansion** (or **development**) of a function and along with it a general method for interpreting an equation between a class symbol and such an expansion. By this technique—widely decried as “mysterious”—Boole had a general and uniform method for obtaining the logical import of any equation in class symbols (i.e., Boolean variables). We give a brief description of this procedure.

If $f(x)$ is an algebraic expression (Boole only considered linear fractional forms in Boolean variables), its expansion is given by

$$f(x) = f(1)x + f(0)(1 - x).$$

Boole “establishes” this identity by assuming $f(x) = ax + b(1 - x)$ and determining a and b by setting $x = 1$ and $x = 0$. Similarly (using x' as an abbreviation for $1 - x$) for an expression in two variables he gives

$$f(x, y) = f(1, 1)xy + f(1, 0)xy' + f(0, 1)x'y + f(0, 0)x'y'.$$

Thus if $f(x, y) = \frac{y}{x}$, then

$$\frac{y}{x} = 1 \cdot xy + 0 \cdot xy' + \frac{0}{0}x'y' + \frac{1}{0}x'y.$$

(We follow Boole's custom of putting the $1/0$ term last.) Boole's argument for equating a function with its expansion is faulty in that it assumes (without justification) that any such function is linear in its Boolean variables. To continue, Boole's rule for interpreting an equation $w = f(x, y, \dots)$ is to equate w to the sum of those terms in the expansion of $f(x, y, \dots)$ which have 1 as their coefficient plus an indefinite multiple of the sum of those terms having $0/0$ as their coefficient and, as an independent condition, the sum of those terms having $1/0$ as coefficient is equated to 0. The reasons Boole gives for his rule are weak and unconvincing and we shall not reproduce them. As an example of its use, the interpretation for the equation $w = y/x$ is that the class w consists of all of xy , none of xy' , an indefinite amount (some, none, or all) of $x'y'$ and, independently, $x'y$ is set equal to 0. The correctness of this as an equivalent to $xw = y$ is easily seen in FIGURE 1.

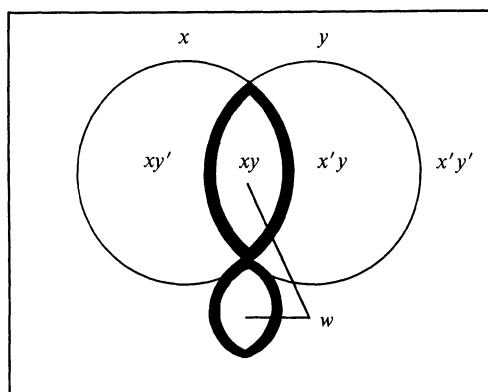


FIGURE 1.

$xw = y$
equivalent to

$$\begin{cases} x'y = 0 \\ w = xy + vx'y', v \text{ indefinite.} \end{cases}$$

In the *Laws of Thought* there are lots of worked-out examples, all with correct conclusions. One is intrigued by this fact and wonders how such a process involving “division” of class expressions, and using entities such as $0/0$ and $1/0$, can be meaningful. We venture an explanation for this in our next section.

Division in Boolean Algebras

In the arithmetic of integers the quotient of m by n exists if $n \neq 0$ and m is a multiple of n . Thus for integers the operation of division is a restricted one and is not always possible. The process of extending the integers to the more extensive structure of the field of rationals is well known: one defines the rationals as equivalence classes of ordered pairs of integers (here ordered pairs = fractions) and with appropriate definitions of addition, multiplication, zero and unit, one has a new structure in which division is possible without exception (save for 0). This structure, moreover, contains a substructure isomorphic to the integers, so that one can still do integer arithmetic within the rationals.

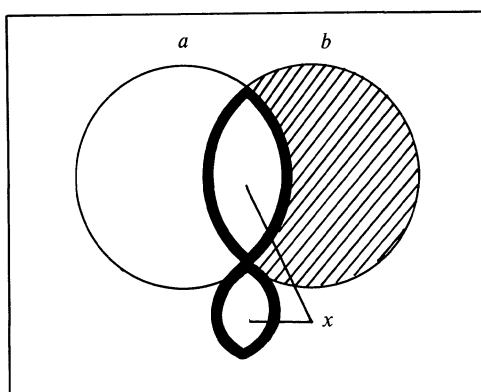


FIGURE 2

Consider now in a Boolean algebra $\mathbf{B}$ the equation

$$ax = b, \quad (1)$$

where a and b are given and x is unknown. It is a simple exercise in Boolean algebra to show that this equation has a solution for x if and only if b is included in a , i.e., if and only if $a'b = 0$. Thus, as with the integers, quotients do exist for some pairs of elements of $\mathbf{B}$. However, unlike the integers, such a quotient need not be unique. As we have already seen in illustrating Boole's interpretive procedure, a solution x for $ax = b$ will be equal to ab plus any amount of $a'b'$ (see FIGURE 2); or, to say it in other words, any two members of the solution set for (1) will differ by a Boolean multiple of (i.e., a part of) $a'b'$. We now show how to “factor out” the differences so as to have a unique quotient (if any exist).

To say that any two solutions of (1) differ by a multiple of $a'b'$ is to say that the solution set is a residue class in the factor ring $\mathbf{B}/(a'b')$, where $(a'b')$ is the principal ideal generated by $a'b'$ (= set of classes included in $a'b'$). If in the factor ring $\mathbf{B}/(a'b')$ we denote by

$$[r] = r + (a'b')$$

the residue class determined by an element r of $\mathbf{B}$, then the mapping $r \rightarrow [r]$ is a homomorphism of rings and hence (1) implies that

$$[a][x] = [b] \quad (2)$$

holds in the ring $\mathbf{B}/(a'b')$. Conversely, (2) implies (1). For if, for $v_1, v_2, v_3 \in \mathbf{B}$, we have

$$(a + v_1 a' b')(x + v_2 a' b') = b + v_3 a' b',$$

then by multiplying this equation through by $(a' b')' (= a \cup b)$ and using simple Boolean identities, one readily obtains (1). Thus the problem of solving for x in (1) is equivalent to finding $[x]$ satisfying (2) in the reduced Boolean algebra $\mathbf{B}/(a' b')$. If there is a solution, say $[x] = [p]$, then, on using the fuller notation for a residue class, this solution is

$$[x] = p + (a' b'). \quad (3)$$

In order to better bring out the relation between what we are doing with Boole's procedure, we now alter the customary notation for a principal ideal in a ring and write $\frac{0}{0} a' b'$ in place of $(a' b')$ so that (3) becomes

$$[x] = p + \frac{0}{0} a' b' \quad (4)$$

which, in terms of elements of $\mathbf{B}$, says

$$x = p + v a' b' \quad (5)$$

where v is an element of $\mathbf{B}$. By going over to $\mathbf{B}/\frac{0}{0} a' b'$ we are now at a stage similar to solving an equation $mx = n$ with m and n integers, where only under limited circumstances does a (unique) quotient exist. We would like the greater algorithmic freedom that one has when working with rationals where division is unrestricted. This is essential if we are to reach our goal of justifying Boole's unusual technique, which involves solving equations such as

$$Ew = F$$

for the unknown w with E and F polynomials in variables $x, y, z, \dots$. To this end we turn to the problem of extending a Boolean algebra by the introduction of quotients.

Extending a Boolean Algebra with Quotients

In contemplating the introduction of quotients and the operation of division into a Boolean algebra, one is confronted with the difficulty that the usual definitions of addition and multiplication of two fractions, namely

$$\begin{aligned} \frac{r_1}{s_1} + \frac{r_2}{s_2} &= \frac{r_1 s_2 + s_1 r_2}{s_1 s_2} \\ \frac{r_1}{s_1} \cdot \frac{r_2}{s_2} &= \frac{r_1 r_2}{s_1 s_2}, \end{aligned}$$

involves the product of the two denominators. In the case of the integral domain of integers, where there are no divisors of zero, if s_1 and s_2 are not 0, then neither is the product $s_1 s_2$. However in the case of Boolean algebra, where $aa' = 0$ for any a , every nonzero element is a divisor of zero. Thus no element of a Boolean algebra can function with complete freedom as a denominator.

Resolution of the difficulty (though not without a price) comes by suitably restricting the denominators to being in a multiplicative set: we say S is a **multiplicative set** if

- (i) $0 \notin S$ and
- (ii) $s_1, s_2 \in S \Rightarrow s_1 s_2 \in S$ for any s_1, s_2 .

For any commutative ring R with unit, by restricting the formation of fractions to those having denominators coming from a multiplicative subset S of R , one can alleviate the aforementioned difficulty. But now the customary definition of the equivalence of two fractions $\left(\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \text{ iff } r_1 s_2 - s_1 r_2 = 0 \right)$ has to be modified to

$$\frac{r_1}{s_1} \sim \frac{r_2}{s_2} \text{ iff for some } s \in S, s(r_1 s_2 - s_1 r_2) = 0. \quad (6)$$

Denote by $\left[\frac{r}{s}\right]$ the equivalence class determined by a pair of elements $r, s \in R, s \in S$, and by RS^{-1} the set of all such equivalence classes. Then with operators $+$ and $\times$ defined by

$$\begin{aligned}\left[\frac{r_1}{s_1}\right] + \left[\frac{r_2}{s_2}\right] &= \left[\frac{r_1 s_2 + s_1 r_2}{s_1 s_2}\right], \\ \left[\frac{r_1}{s_1}\right] \left[\frac{r_2}{s_2}\right] &= \left[\frac{r_1 r_2}{s_1 s_2}\right],\end{aligned}$$

and with $\left[\frac{0}{1}\right]$ and $\left[\frac{1}{1}\right]$ as the zero and unit, one can show that RS^{-1} is a ring—the ring of quotients of R by S [12]. And now here is the price we have to pay in order to have quotients in a general ring: in the resulting ring of quotients not every member of R can be a denominator, nor need the ring of quotients be a field; moreover, although the mapping $h: R \rightarrow RS^{-1}, h(r) = \left[\frac{r}{1}\right]$, is a homomorphism of rings, it need not be injective—as would be the case for the rationals.

To apply this theory to a Boolean algebra (which we take in the form of a ring with $+$ standing for symmetric difference) one chooses as a denominator set any filter. If e is a nonzero element of a Boolean algebra $\mathbf{B}$, then the filter determined by e consists of all elements which include e . We may conveniently designate this set by $e \cup \mathbf{B}$. Clearly $e \cup \mathbf{B}$ is a multiplicative set and thus $\mathbf{B}(e \cup \mathbf{B})^{-1}$ is a ring of quotients of $\mathbf{B}$ by $e \cup \mathbf{B}$. As the usual notation for quotients is here insufficient in that it contains no indication of the denominator set, let us use $\left(\frac{r}{s}\right)_e$ where the e suffices to indicate the denominator set $e \cup \mathbf{B}$.

There are a few results about Boolean quotients to which we will refer:

THEOREM 1. *For any element a in the denominator set $e \cup \mathbf{B}$*

$$\begin{aligned}\text{(i)} \quad \left(\frac{ab}{a}\right)_e &= \left(\frac{b}{1}\right)_e, \\ \text{(ii)} \quad \left(\frac{b}{a}\right)_e &= \left(\frac{ab}{1}\right)_e.\end{aligned}$$

Proof. Both results are immediate consequences of definition (6); (i) by virtue of $e(1 \cdot ab - a \cdot b) = 0$, and (ii) by $a(b \cdot 1 - a \cdot ab) = 0$.

THEOREM 2. *The factor ring $\mathbf{B}/(e')$ and the ring of quotients $\mathbf{B}(e \cup \mathbf{B})^{-1}$ are, under the mapping $b + (e') \rightarrow \left(\frac{b}{1}\right)_e$, isomorphic structures.*

Proof. [2, p. 34].

We now show how to reproduce, with due mathematical rigor, Boole's solution of a Boolean equation by division and expansion, and to justify his interpretation for the algebraic solution. Since the solution of a Boolean equation is directly obtainable in Boolean algebra, our procedure, which uses a number of new structures, will appear complicated and circuitous. We emphasize, however, that it is the reproducing of Boole's method, not the solution of the equation, which is of interest.

Consider an equation

$$ax = b \tag{7}$$

where $a (\neq 0)$ and b are elements of a Boolean algebra $\mathbf{B}$. As we have seen above, this equation has a solution for x in $\mathbf{B}$ if and only if the equation

$$[a][x] = [b] \tag{8}$$

has a solution for $[x]$ in $\mathbf{B}^* = \mathbf{B}/\frac{0}{0}a'b'$, where $\frac{0}{0}a'b'$ is the principal ideal generated by $a'b'$ and $[r]$ designates the residue class modulo $\frac{0}{0}a'b'$ determined by r ; moreover, if $[x] = [p]$ is the

solution of (8), then the solution set for (7) is given by

$$x = p + va'b', \quad v \text{ ranging over all elements of } \mathbf{B}.$$

Suppose now that (8) has a solution. In the ring of quotients $\mathbf{Q} = \mathbf{B}^*([a] \cup \mathbf{B}^*)^{-1}$ we have, from (8),

$$\left(\frac{[a][x]}{[a]} \right)_a = \left(\frac{[b]}{[a]} \right)_a, \quad (9)$$

from which, by cancellation (Theorem 1(i)),

$$\left(\frac{[x]}{[1]} \right)_a = \left(\frac{[b]}{[a]} \right)_a. \quad (10)$$

This is what for us corresponds to Boole's $x = b/a$. Using Theorem 1(ii) and the fact that the mapping $r \rightarrow [r]$ is a homomorphism to replace $[a][b]$ by $[ab]$, we obtain from (10) that

$$\left(\frac{[x]}{[1]} \right)_a = \left(\frac{[ab]}{[1]} \right)_a \quad (11)$$

in the ring $\mathbf{Q}$. We now go over to the ring $\mathbf{B}^*/([a'])$ which, by Theorem 2, is isomorphic to $\mathbf{Q}$. But first we note that $[a'] = [a]$, and that $[a'] = [a'b]$ since a' and $a'b$ differ by $a'b'$ and thus determine the same residue class modulo $a'b'$. Hence the ideal $([a'])$ is $([a'b])$. We change the usual notation and represent this ideal by $\frac{1}{0}a'b$. (Note that $\frac{1}{0}a'b$ is an ideal of $\mathbf{B}^*$, whereas $\frac{0}{0}a'b'$ is an ideal of $\mathbf{B}$.) Using " $\approx$ " to indicate the relation of isomorphism between elements of $\mathbf{B}^*([a] \cup \mathbf{B}^*)^{-1}$ and $\mathbf{B}^*/([a'])$ we have by Theorem 2

$$\left(\frac{[b]}{[a]} \right)_{[a]} \approx [ab] + \frac{1}{0}a'b$$

or, in fuller notation,

$$\left(\frac{[b]}{[a]} \right)_a \approx 1 \cdot ab + 0 \cdot ab' + \frac{0}{0}a'b' + \frac{1}{0}a'b \quad (12)$$

which is what for us corresponds to Boole's

$$\frac{b}{a} = 1 \cdot ab + 0 \cdot ab' + \frac{0}{0}a'b' + \frac{1}{0}a'b.$$

Since he also has $x = b/a$, he can then write

$$x = 1 \cdot ab + 0 \cdot ab' + \frac{0}{0}a'b' + \frac{1}{0}a'b,$$

whereas we can only equate the isomorphic images of the members of (11) to obtain

$$\begin{aligned} [x] + \frac{1}{0}a'b &= [ab] + \frac{1}{0}a'b \\ &= 1 \cdot ab + 0 \cdot ab' + \frac{0}{0}a'b' + \frac{1}{0}a'b. \end{aligned} \quad (13)$$

Now for the Boolean interpretation (i.e., in terms of $\mathbf{B}$) of (13). If $ax = b$ then $a'b = a'ax = 0$, so that

$$\frac{1}{0}a'b = ([a'b]) = ([0])$$

is the zero ideal of $\mathbf{B}^*$. Thus from (13)

$$[x] = [ab] = 1 \cdot ab + 0 \cdot ab' + \frac{0}{0}a'b'$$

and so by (5),

$$x = ab + va'b' \quad v \in \mathbf{B}.$$

Consequently (7) implies, through (13), that

$$\begin{cases} x = ab + va'b' \\ a'b = 0 \end{cases} \quad v \in \mathbf{B} \quad (14)$$

which is precisely what Boole gives as his interpretation for

$$x = 1 \cdot ab + 0 \cdot ab' + \frac{0}{0} a'b' + \frac{1}{0} a'b.$$

Boole didn't consider a converse argument (i.e., from (14) to (7)) since it was unnecessary from his point of view. For him the processes of "common" algebra were applicable to logical equations and hence b/a would obviously be the correct solution to $ax = b$. The only problem for him was to obtain the logical meaning of $x = b/a$.

Summary

We have briefly described the nascent abstract algebra ideas within which Boole originated his algebra of logic. While he never made his algebra fully explicit, we inferred that what he did use was, if clarified, a commutative ring with unit, without nilpotents, and having idempotents which stood for classes. By thus hewing closely to "common" algebra Boole could use familiar procedures and techniques. He did not realize that class calculus needed only the idempotents and operations closed with respect to them (i.e., Boolean algebra). Instead he used the ring operations and, in particular, its addition, which is not closed with respect to idempotency. Boole also freely used division to solve equations, introducing then a special, not clearly explained or justified, method for extracting the logical content of the resulting quotient. To explain this we found it necessary to introduce additional structures—factor rings, and rings of quotients for such rings.

We have alluded to the possibility of applications for Boole's algebra—to multisets and to pseudo-Boolean functions. As an additional possibility we would also like to mention that in his *Laws of Thought* Boole uses his peculiar expansions with $1/0$ to present an original approach to conditional probability [2, p. 195].

References and Readings

- For matters relating to the history of algebra relevant to this paper, we recommend [5] and [7], and for the history of logic, [5]. For an alternative to our justification of Boole's methods, one may consult [11, pp. 191–197].
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