Interleaving Schemes for Multidimensional Cluster Errors

Mario Blaum
IBM Research Division
650 Harry Road
San Jose, CA 95120, USA
blaum@almaden.ibm.com

Jehoshua Bruck *
California Institute of Technology
Mail Stop 136-93
Pasadena, CA 91125, USA
bruck@paradise.caltech.edu

Abstract

We present 2 and 3-dimensional interleaving techniques for correcting 2 and 3-dimensional bursts (or clusters) of errors, where a cluster of errors is characterized by its area or volume. A recent application of correction of 2-dimensional clusters appeared in the context of holographic storage. Our main contribution is the construction of efficient 2 and 3-dimensional interleaving schemes. The schemes are based on arrays of integers with the property that every connected component of area or volume t consists of distinct integers (we call these t-interleaved arrays). In the 2-dimensional case, our constructions are optimal in the sense that they contain the smallest possible number of distinct integers, hence minimizing the number of codes required in an interleaving scheme.

Codewords: Error-correcting codes, bursts, interleaving, 2 and 3-dimensional interleaving, clusters, 2-dimensional bursts.

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1 Introduction

A one-dimensional burst error of length t is a set of errors that are confined to t consecutive locations [14]. In this paper, we generalize the concept of the one-dimensional burst to two and three dimensions by considering the connected area or volume, say t, containing the errors. Most 2-dimensional burst error-correcting codes that have been studied in the literature so far consider burst errors of a given rectangular shape, say $t_1 \times t_2$ [1, 3, 4, 7, 10, 11, 12]. However, there are also papers that study other shapes as well. For instance, in [2], the authors study "circular" type of bursts. In [6, 9, 15], the authors consider metrics given by the rank of the array: a particular case, is the correction of "criss-cross" type of errors. Metrics for different channels, including 2-dimensional clusters, are presented in [8]. A recent application of correction of 2-dimensional clusters appeared in the context of holographic storage [13].

The most common approach to deal with one-dimensional bursts is using interleaving schemes. The idea is to implement a number of separate codes on consecutive symbols. For example, to deal with correction of bursts of length 3 one can use 3 different 1-error correcting codes that encode an interleaved sequence as follows:

$$123123123123123123123123123123\ldots$$

Here, 1,2 and 3 correspond to the first, second and third code, respectively. This straightforward interleaving scheme requiring t different codes for bursts of length t is optimal in the sense that there is no other interleaving scheme that can correct a burst of length up to t that requires less than t different codes.

However, in the 2-dimensional case, it is not obvious how to interleave a minimal number of codes such that any cluster of area t can be corrected. Our main contribution is the construction of efficient 2 and 3-dimensional interleaving schemes. In the 2-dimensional case, our constructions are optimal in the sense that they contain the smallest possible number of distinct codes. We note here that a related construction with the constraint that the area has a rectangular shape was presented in [5].

Next we formalize the problem of constructing 2-dimensional interleaving schemes. The 3-dimensional case will be presented in Section 3.

Definition 1.1 We say that an element (i, j) in a 2-dimensional array is *connected* to elements $(i+1, j)$, $(i-1, j)$, $(i, j+1)$ and $(i, j-1)$, provided those elements exist.

Definition 1.2 A *path* of length n from E_0 to E_n in a 2-dimensional array is a set of $n+1$ elements $\{E_i \mid 0 \leq i \leq n\}$ such that for every $0 \leq i < n$, element E_i is connected to element E_{i+1}.

Definition 1.3 We say that a set of t elements in a 2-dimensional array is a *cluster* of size t, if any two elements in the cluster belong in a path contained in the set.

The concept of a cluster of size t generalizes in two dimensions the concept of a burst of size t in one dimension. The same idea can be generalized to multiple dimensions (see Section 3).

Example 1.1 The 1's in the array below constitute a cluster of size 7.

0	0	0	0	0	0	0
0	1	1	1	0	0	0
0	0	1	1	1	0	0
0	0	0	1	0	0	0

Definition 1.4 Let A be a 2-dimensional array of integers, namely, the elements of the array are labeled by integers. We say that A is *t-interleaved* if every cluster of size t in A consists of t distinct integers. The *degree of interleaving* of the array is the number of distinct integers it contains.

Notice that, if the integers represent different codes (like in the one-dimensional case), then codes distributed in a t-interleaved array can correct any cluster of size up to t (or more than one cluster, depending on the error-correcting capability of the codes being used).

Example 1.2 The following array is 3-interleaved with degree of interleaving 5:

0	1	2	3	4	0	1
3	4	0	1	2	3	4
1	2	3	4	0	1	2
4	0	1	2	3	4	0
2	3	4	0	1	2	3

Our goal is to construct t-interleaved arrays with minimal degree of interleaving. Notice that in the one-dimensional case, the minimal degree of interleaving t coincides with the size of the burst we want to correct. This is not the case in the 2-dimensional case, as we will see in the sequel.

In the next section we present optimal 2-dimensional interleaving schemes. In Section 3 we generalize our methods to three (and more) dimensions.

2 Two-Dimensional Interleaving

In this section we present two optimal constructions for t-interleaved arrays. We start by presenting lower bounds on the degree of interleaving of t-interleaved arrays.

2.1 Lower Bounds

Theorem 2.1 Let A be a t-interleaved array. Then

1. If t is even, then the degree of interleaving of A is at least $\frac{t^2}{2}$.
2. If t is odd, then the degree of interleaving of A is at least $\frac{t^2+1}{2}$.

Proof: The idea of the proof is to take a t-interleaved array and to consider a 2-dimensional "sphere" in the array, of size $\frac{t^2}{2}$ when t is even and size $\frac{t^2+1}{2}$ when t is odd. Then we show that any two elements in the sphere must be distinct.

In particular, for every t, we define 2-dimensional spheres and we denote them by $B_2(t)$. $B_2(t)$ is defined inductively for the odd and even cases.

Consider an array. The sphere $B_2(1)$ is a single element in the array. The sphere $B_2(2)$ is a 1×2 subarray. The sphere $B_2(t+2)$ is constructed from $B_2(t)$ by adding all the elements in the array that are connected to the boundary of $B_2(t)$.

It is straightforward to verify that $B_2(t)$ consists of $\frac{t^2}{2}$ elements for t even and $\frac{t^2+1}{2}$ elements for t odd.

For example, the first six $B_2(t)$'s are as follows ($B_2(t)$ is labeled by $\clubsuit$):

- $t = 1$

♣

- $t = 2$

♣	♣

- $t = 3$

	♣	
♣	♣	♣
	♣	

- $t = 4$

	♣	♣	
♣	♣	♣	♣
	♣	♣	

- $t = 5$

		♣		
	♣	♣	♣	
♣	♣	♣	♣	♣
	♣	♣	♣	
		♣		

- $t = 6$

		♣	♣		
	♣	♣	♣	♣	
♣	♣	♣	♣	♣	♣
	♣	♣	♣	♣	
		♣	♣		

Next we prove that for every t, any two elements of $B_2(t)$ are connected by a path of length at most $t-1$. The proof is by induction. Clearly, the claim is true for $t=1$ and $t=2$. Now we assume that the claim is true for t and we prove it for $t+2$. Notice that by the construction, $B_2(t)$ is contained in $B_2(t+2)$. Let i and j be two arbitrary elements of $B_2(t+2)$. If both are also elements of $B_2(t)$ then by the induction hypothesis there is a path of length at most $t-1$ between them. Otherwise, by the construction of $B_2(t)$, an element in $B_2(t+2)$ that is not in $B_2(t)$ is connected to an element in $B_2(t)$. Hence, there is a path of length at most $t+1$ between i and j, proving the induction.

Since $B_2(t)$ is contained in a t-interleaved array, it must consist of distinct elements. Therefore, the degree of interleaving of the array is at least the number of elements of $B_2(t)$. □

2.2 Constructions

Next we present constructions of t-interleaved arrays of optimal size, namely, they match the lower bounds described above.

First we describe an interleaving scheme that we call the *toroidal* interleaving scheme.

Construction 2.1 Consider a 2-dimensional array and an integer m. Label the coordinates of the array toroidally on m, i.e., the coordinates are given by (x, y), where x and y are taken modulo m. Let b be relatively prime with m. Then, for each a modulo m, the coordinates $(i, a+ib)$ (taken modulo m) are assigned the same number a.

Example 2.1 Assume that we have a 4×6 array. Taking $m=2$ and $b=1$, Construction 2.1 gives the following labeling of the array:

0	1	0	1	0	1
1	0	1	0	1	0
0	1	0	1	0	1
1	0	1	0	1	0

It is easy to verify that the array above is 2-interleaved.

Similarly, if we consider a 5×10 array for $m = 5$ and $b = 3$, we obtain

0	1	2	3	4	0	1	2	3	4
2	3	4	0	1	2	3	4	0	1
4	0	1	2	3	4	0	1	2	3
1	2	3	4	0	1	2	3	4	0
3	4	0	1	2	3	4	0	1	2

The reader can verify that the array above is 3-interleaved.

As we can see in Example 2.1, given an array labeled by Construction 2.1, in order to find if the array is t interleaved, it is enough to consider the $m \times m$ array obtained by the construction. The labeling of the whole array is obtained by tiling it with the $m \times m$ array.

Definition 2.1 The *Lee distance* between two elements in a torus is the length of the shortest path they belong to (for example, two adjacent elements are at Lee distance 1). The *Lee weight* of an element in a torus is the Lee distance between the element and (0,0). The minimum Lee distance of a set of elements is the minimum of the Lee distance between all the pairs of elements in the set.

The following lemma gives a method for finding t in Construction 2.1.

Lemma 2.1 Consider Construction 2.1 with parameters m and b. Let t be the minimum Lee distance in the $m \times m$ torus between two coordinates labeled with the same number. Then, the array is t-interleaved. In particular, it is enough to consider the minimum Lee distance between those coordinates labeled with $a = 0$, i.e., between the coordinates (i, ib), $0 \leq i \leq m - 1$.

Proof: Consider a cluster of size at most t. Take any two coordinates in the cluster. There is a path from one to the other of length at most $t-1$ which is contained in the cluster. Therefore, the Lee distance between the two coordinates is at most $t-1$. By hypothesis, they cannot have the same label, proving the claim. □

In other words, it is enough to analyze the set $\{(i, ib) \; : \; 0 \leq i \leq m-1\}$ and find its minimum Lee distance t in order to determine if the array with the toroidal labeling defined by m and b is t-interleaved.

The next theorem is our main result in this section.

Theorem 2.2 Let t be an odd integer, $m = \frac{t^2+1}{2}$, and $b = t$. Then, Construction 2.1 with parameters m and b gives a t-interleaved array.

Proof: According to Lemma 2.1, it is enough to prove that the set $\{(i, it) \; : \; 0 \leq i \leq m-1\}$ has minimum Lee distance at least t, where the coordinates are taken modulo $\frac{t^2+1}{2}$.

The case $t = 1$ is trivial, so assume that $t \geq 3$. Since t is an odd integer, either $t = 4j-1$ or $t = 4j+1$, where $j \geq 1$. We study the case $t = 4j-1$ only, the other one is proven similarly.

Without loss of generality, it is enough to measure the Lee distance between (i, it) and $(0,0)$, since the set $\{(i, it) \; : \; 0 \leq i \leq m-1\}$ is linear (i.e., it is enough to find the minimal Lee weight of the set).

Notice that the Lee weight of (i, it) is given by $\min\{i, -i\} + \min\{it, -it\}$, where all the values are taken modulo $\frac{t^2+1}{2}$.

It is enough to consider those i's such that $1 \leq i \leq t-1$ or $\frac{t^2+1}{2} - (t-1) \leq i \leq \frac{t^2-1}{2}$, otherwise either $i \geq t$ or $-i \geq t$. Moreover, since $(-i, -it)$ has the same Lee weight as (i, it), it is enough to assume that $1 \leq i \leq t-1$. Also, notice that $\frac{t^2+1}{2} = 8j^2 - 4j + 1$. There are four cases:

$1 \leq i \leq j-1:$ Notice that $it \leq j(4j-1) \leq 4j^2 - 2j \leq \frac{t^2+1}{4}$, so $\min\{it, -it\} = it$. Therefore, the Lee weight of (i, it) is given by $i + it = i(t+1) \geq t+1$, proving the claim.

$j \le i \le 2j-1:$ These conditions imply that $\frac{t^2+1}{4} \le it < \frac{t^2+1}{2}$. Thus, $\min\{it, -it\} = \frac{t^2+1}{2} - it$. The Lee weight of (i, it) is then given by

$$\frac{t^2+1}{2} - i(t-1) \quad \ge \quad 8j^2 - 4j + 1 - (2j-1)(4j-2) = 4j - 1 = t,$$

proving the claim.

$2j \le i \le 3j-1:$ These conditions imply that $\frac{t^2+1}{2} \le it \le \frac{3t^2}{4}$. Thus, $\min\{it, -it\} = it - \frac{t^2+1}{2}$. The Lee weight of (i, it) is then given by

$$i(t+1) - \frac{t^2+1}{2} \quad \ge \quad 8j^2 - 8j^2 + 4j - 1 = 4j - 1 = t,$$

proving the claim.

$3j \le i \le 4j-2:$ These conditions imply that $\frac{3t^2}{4} \le it \le t^2$. Thus, $\min\{it, -it\} = t^2 + 1 - it$. The Lee weight of (i, it) is then given by

$$t^2 + 1 - i(t-1) \quad \ge \quad 16j^2 - 8j + 2 - (4j-2)^2 = 8j - 2 = 2t,$$

proving the claim.

The case $t = 4j+1$ is proven similarly. □

The next theorem gives an analogous result for t even. The proof is similar to that of Theorem 2.2, and we omit it here.

Theorem 2.3 Let t be an even integer, $m = \frac{t^2}{2}$, and $b = t - 1$. Then, Construction 2.1 with parameters m and b gives a t-interleaved array.

Example 2.2 Consider the case $t = 4$. According to Theorem 2.3, $m = 8$ and $b = 3$. Therefore, tiling an array with the following 8×8 array gives a 4-interleaved array:

0	1	2	3	4	5	6	7
5	6	7	0	1	2	3	4
2	3	4	5	6	7	0	1
7	0	1	2	3	4	5	6
4	5	6	7	0	1	2	3
1	2	3	4	5	6	7	0
6	7	0	1	2	3	4	5
3	4	5	6	7	0	1	2

For $t = 5$, according to Theorem 2.2, $m = 13$ and $b = 5$. Therefore, tiling an array with the following 13×13 array gives a 5-interleaved array:

0	1	2	3	4	5	6	7	8	9	10	11	12
8	9	10	11	12	0	1	2	3	4	5	6	7
3	4	5	6	7	8	9	10	11	12	0	1	2
11	12	0	1	2	3	4	5	6	7	8	9	10
6	7	8	9	10	11	12	0	1	2	3	4	5
1	2	3	4	5	6	7	8	9	10	11	12	0
9	10	11	12	0	1	2	3	4	5	6	7	8
4	5	6	7	8	9	10	11	12	0	1	2	3
12	0	1	2	3	4	5	6	7	8	9	10	11
7	8	9	10	11	12	0	1	2	3	4	5	6
2	3	4	5	6	7	8	9	10	11	12	0	1
10	11	12	0	1	2	3	4	5	6	7	8	9
5	6	7	8	9	10	11	12	0	1	2	3	4

Theorems 2.2 and 2.3 give optimal interleaving schemes for any t, since they meet the lower bound given by Theorem 2.1.

Next we present an optimal construction for t even.

Construction 2.2 Let t be even. Let $C_1(t)$ be the $\frac{t}{2} \times \frac{t}{2}$ array labeled by the integers $\{j : 0 \leq j \leq \frac{t^2}{4} - 1\}$ and $C_2(t)$ be the $\frac{t}{2} \times \frac{t}{2}$ array labeled by the integers $\{j : \frac{t^2}{4} \leq j \leq \frac{t^2}{2} - 1\}$.

The t interleaved array $A_2(t)$ consists of the chess-board-like tiling using the arrays $C_1(t)$ and $C_2(t)$.

Example 2.3 Let $t = 6$. Then

$C_1(6) =$

0	1	2
3	4	5
6	7	8

$C_2(6) =$

9	10	11
12	13	14
15	16	17

and

$A_2(6) =$

0	1	2	9	10	11	0	1	2	9	10	11
3	4	5	12	13	14	3	4	5	12	13	14
6	7	8	15	16	17	6	7	8	15	16	17
9	10	11	0	1	2	9	10	11	0	1	2
12	13	14	3	4	5	12	13	14	3	4	5
15	16	17	6	7	8	15	16	17	6	7	8

Theorem 2.4 For every even t, the arrays $A_2(t)$ in Construction 2.2 are t-interleaved.

Proof: The proof follows by observing that a path connecting any two elements with the same label, say 0 in C_1 arrays, must go through a C_2 array. Hence, it is of length at least t. $\square$

3 Three Dimensional Interleaving

In this section we extend the results of the previous section to the case of three dimensions. The results can be further extended to higher dimensions, but we will not do it here.

We briefly adapt some of the definitions given in the introduction.

Definition 3.1 We say that an element (i_1, i_2, i_3) in a 3-dimensional array is *connected* to elements $(i_1 + 1, i_2, i_3)$, $(i_1 - 1, i_2, i_3)$, $(i_1, i_2 + 1, i_3)$, $(i_1, i_2 - 1, i_3)$, $(i_1, i_2, i_3 + 1)$ and $(i_1, i_2, i_3 - 1)$, provided those elements exist.

Notice that the definition above can be trivially extended to multiple dimensions. In a k-dimensional array, an element is in general connected to $2k$ elements in the array.

Definition 3.2 A *path* of length n from E_0 to E_n in a 3 (multi)-dimensional array is a set of $n+1$ elements $\{E_i \,|\, 0 \leq i \leq n\}$ such that for every $0 \leq i < n$, element E_i is connected to element E_{i+1}.

Definition 3.3 We say that a set of t elements in a 3 (multi)-dimensional array is a *cluster* of size t, if any two elements in the cluster belong in a path contained in the set.

Definition 3.4 Let A be a 3 (multi)-dimensional array of integers, namely, the elements of the array are labeled by integers. We say that A is *t-interleaved* if every cluster of size t in A consists of t distinct integers. The *degree of interleaving* of the array is the number of distinct integers it contains.

As in the previous section, we start with lower bounds.

3.1 Lower Bounds

Theorem 3.1 Let A be a t-interleaved 3-dimensional array. Then

1. If t is even, then the degree of interleaving of A is at least $\frac{t^3+2t}{6}$.
2. If t is odd, then the degree of interleaving of A is at least $\frac{t^3+5t}{6}$.

Proof: As in Theorem 2.1, we take a t-interleaved array and consider a 3-dimensional "sphere" in the array, which we will see that it has size $\frac{t^3+2t}{6}$ when t is even and size $\frac{t^3+5t}{6}$ when t is odd. Any two elements in the sphere must be distinct.

For every t, we define 3-dimensional spheres and we denote them by $B_3(t)$. $B_3(t)$ is defined inductively for the odd and even cases.

Consider a 3-dimensional t-interleaved array. The sphere $B_3(1)$ is a single element in the array. The sphere $B_3(2)$ is a 1×2 subarray. The sphere $B_3(t+2)$ is constructed from $B_3(t)$ by adding all the elements in the array that are connected to the boundary of $B_3(t)$.

An easy counting argument shows that the cardinality of $B_3(t)$ is $\frac{t^3+2t}{6}$ when t is even and $\frac{t^3+5t}{6}$ when t is odd.

Since the array is t-interleaved, it is shown that the elements in any sphere $B_3(t)$ must be different similarly to Theorem 2.1. $\square$

For instance, the first 6 $B_3(t)$'s are as follows:

- $t = 1$

1

- $t = 2$

1	1

- $t = 3$

	1	
1	3	1
	1	

- $t = 4$

	1	1	
1	3	3	1
	1	1	

- $t = 5$

		1		
	1	3	1	
1	3	5	3	1
	1	3	1	
		1		

- $t = 6$

		1	1		
	1	3	3	1	
1	3	5	5	3	1
	1	3	3	1	
		1	1		

The numbers above represent a 3-dimensional "sphere": wherever we see numbers, we have a 2-dimensional projection of the 3-dimensional sphere over the plane of the paper. These projections are 2-dimensional spheres as described in the previous section. The numbers indicate how many symbols we have in each level. For instance, for $t = 3$ above, the 3 indicates that there is one symbol above and one symbol below the plane of the paper. The 5 indicates that there are two above and two below, and so on.

3.2 Constructions

Next we provide some constructions that give upper bounds on the degree of interleaving in 3-dimensions of a t-interleaved array. In most cases, we cannot reach the lower bound given by Theorem 3.1, as in the 2-dimensional case. We do not know if the lower bound is tight.

We describe next a toroidal interleaving scheme that is a generalization of the one given in the previous section.

Construction 3.1 Consider a 3-dimensional array and an integer m. Label the coordinates of the array toroidally on m, i.e., the coordinates are given by (x, y, z), where x, y and z are taken modulo m. Let at least b or c be relatively prime with m. Then, for each a modulo m, the coordinates $(i, j, a + ib + jc)$ (taken modulo m) are assigned the same number a.

Construction 3.1 gives a 3-dimensional interleaving scheme with degree of interleaving m. The array is t-interleaved, where t is the minimum (Lee) distance in the torus between coordinates with the same number a. Since coordinates with different numbers a are essentially translations from each other, without loss of generality, in order to measure the minimum (Lee) distance, it is enough to consider $a = 0$.

Example 3.1 Consider Construction 3.1 with $m = 7$, $b = 2$ and $c = 3$. The coordinates labeled by 0 are $(i, j, 2i + 3j)$, where everything is taken modulo 7. Explicitly, they are,

$(0,0,0)$	$(0,1,3)$	$(0,2,6)$	$(0,3,2)$	$(0,4,5)$	$(0,5,1)$	$(0,6,4)$
$(1,0,2)$	$(1,1,5)$	$(1,2,1)$	$(1,3,4)$	$(1,4,0)$	$(1,5,3)$	$(0,6,6)$
$(2,0,4)$	$(2,1,0)$	$(2,2,3)$	$(2,3,6)$	$(2,4,2)$	$(2,5,5)$	$(2,6,1)$
$(3,0,6)$	$(3,1,2)$	$(3,2,5)$	$(3,3,1)$	$(3,4,4)$	$(3,5,0)$	$(3,6,3)$
$(4,0,1)$	$(4,1,4)$	$(4,2,0)$	$(4,3,3)$	$(4,4,6)$	$(4,5,2)$	$(4,6,5)$
$(5,0,3)$	$(5,1,6)$	$(5,2,2)$	$(5,3,5)$	$(5,4,1)$	$(5,5,4)$	$(5,6,0)$
$(6,0,5)$	$(6,1,1)$	$(6,2,4)$	$(6,3,0)$	$(6,4,3)$	$(6,5,6)$	$(6,6,2)$

It can be easily verified that the minimum Lee weight of the set above is 3, therefore, the resulting array is 3-interleaved, i.e., every cluster of size 3 has different numbers. Since, for $t = 3$, the lower bound on the degree of interleaving is 7, this construction is optimal.

In order to obtain the maximal value of t from Construction 3.1 we optimize over all possible values of b and c. This gives us upper bounds on the degree of interleaving for a given t. Table 1 presents lower bounds based on Theorem 3.1 and upper bounds based on Construction 3.1 that were obtained by a computer search. We also add in the table values of b and c that optimize the construction (of course, they are not necessarily unique).

We describe next an interleaving scheme that is a generalization of Construction 2.2 given in the previous section.

Construction 3.2 Assume that t is even. The construction is recursive. Assume that we have an interleaving scheme for the case $\frac{t}{2}$ that we call $A(t/2)$. Replace every label in $A(t/2)$ by a $2 \times 2 \times 2$ array C_i, where C_i consists of the 8 integers $\{8i + j | 0 \le j \le 7\}$.

Using an argument similar to the one in Theorem 2.4, we can prove that the array given by Construction 3.2 is t-interleaved.

Example 3.2 Consider Construction 3.2 with $t = 2$ and $m = 2$, namely we consider a $2 \times 2 \times 2$ torus with the first plane being

0	1
1	0

t	Lower bound	Upper bound Construction 3.1	(b,c)	Upper bound Construction 3.2
2	2	2	(1,1)	
3	7	7	(2,3)	
4	12	12	(3,5)	16
5	25	27	(4,10)	
6	38	38	(7,11)	56
7	63	70	(16,25)	
8	88	92	(9,39)	96
9	129	145	(9,61)	
10	170	190	(9,71)	216
11	231	260	(40,94)	
12	292	312	(13,115)	304
13	377	421	(16,182)	
14	462	486	(41,57)	560
15	575	635	(146,274)	
16	688	724	(49,79)	736

Table 1: Lower and upper bounds on the degree of interleaving of 3-dimensional t-interleaved arrays

and the second plane being

1	0
0	1

Using Construction 3.2 for $t = 4$, we replace the 0 by the $2 \times 2 \times 2$ torus consisting of the following two planes:

0	1
2	3

4	5
6	7

and we replace the 1 by the $2 \times 2 \times 2$ torus consisting of the following two planes:

8	9
10	11

12	13
14	15

The degree of interleaving for t using Construction 3.2 is 8 times the degree of interleaving that we had for $t/2$.

Assuming that we use an optimal construction for the case $t/2$ (i.e., a construction meeting the lower bound), we can prove the following:

Lemma 3.1 Assume that we are given an optimal 3-dimensional t-interleaved array. Then, using Construction 3.2 to construct a 3-dimensional $2t$-interleaved array, this array has degree of interleaving $2t$ away from the lower bound when t is even ($6t$ away from the lower bound when t is odd).

Proof: We will prove the lemma for even t. The odd case is proven similarly.

By hypothesis and Theorem 3.1, the degree of the t-interleaved array is:

$$\frac{t^3+2t}{6}.$$

The degree of the $2t$-interleaved array using Construction 3.2 is:

$$\frac{8(t^3+2t)}{6} = \frac{(2t)^3+2(2t)}{6} + 2t.$$

□

We include the degrees of interleaving associated with Construction 3.2 in Table 1. We can see that for $t = 12$, it improves Construction 3.1.

In Construction 3.2 we replace every label by a $2 \times 2 \times 2$ array. This construction can be generalized to include the case of $d \times d \times d$ arrays. However, it is not difficult to see that the choice $d = 2$ provides the smallest degree of interleaving.

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