

Ad hoc wireless networks with noisy links

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Abstract—Models of ad-hoc wireless networks are often based on the *geometric disc abstraction*: transmission is assumed to be isotropic, and reliable communication channels are assumed to exist (apart from interference) between nodes closer than a given distance. In reality communication channels are unreliable and communication range is generally not rotationally symmetric. In this paper we examine how these issues affect network connectivity. Using ideas from percolation theory, we compare networks of geometric discs to other simple shapes, including probabilistic connections, and we find that when transmission range and node density are normalized across experiments so as to preserve the *expected number of connections* (ENC) enjoyed by each node, the discs are the “hardest” shape to connect together. In other words, anisotropic radiation patterns and spotty coverage allow an unbounded connected component to appear at lower ENC levels than perfect circular coverage allows. This indicates that connectivity claims made in the literature using the geometric disc abstraction will in general hold also for the more irregular shapes found in practice.

I. INTRODUCTION

Ad-hoc wireless networks are usually defined as a collection of computers equipped with radio transmitters and communicating in a multi-hop fashion, routing each other’s data packets. Recently, there has been a growing interest in modeling global properties of these networks as a function of the node population, and basic results on connectivity and throughput capacity have been obtained [5] [6] [7] [8]. These works typically assume that nodes are randomly located and that they are able to reliably relay messages to sufficiently close neighbors.

A first stochastic model that exploited these ideas appeared in the 1961 paper of Gilbert’s [3], who studied the problem of multi-hop connectivity of wireless broadcasting stations, establishing the foundations of the theory of Continuum Percolation [10]. In his formulation, points of a two-dimensional Poisson point process represent wireless transmitting stations of range $2r$ and he asks if the system can provide some long-distance communication. He shows the existence of a critical value λ_c for the density of the transmitters, such that, for $\lambda > \lambda_c$, an unbounded connected component of transmitters forms (i.e., the network *percolates*), with probability one, and so long-distance multi-hop communication is possible.

In the past forty years Gilbert’s model has received much attention from mathematicians and physicists and, recently, extensions relevant to wireless network design have also been proposed [1] [2] [14] [15]. In this paper we consider a further extension that is useful to model more realistic, unreliable and non-rotationally symmetric communication channels (see Fig 1).

We consider a general random connection model where each pair of nodes can be connected according to some (probabilis-

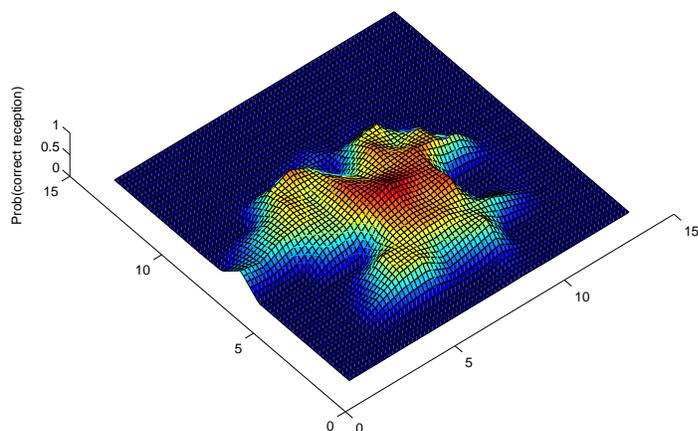


Fig. 1. Real wireless links. The figure depicts the estimated probability distribution for packets to be correctly received in a real wireless ad-hoc network. The experiment was carried out in an open space, using 168 nodes (Berkeley Rene motes running Tiny-OS operating system) placed on the ground and forming a 12×14 grid, with grid spacing of 2 feet. Only one node transmitted broadcast messages, and the figure depicts the fraction of messages received by surrounding nodes. Values for non-grid positions are interpolated. Data is available on-line and was taken from: <http://localization.millennium.berkeley.edu>

tic) function of their (random) position. Consider, for example, placing nodes at random on the plane and assuming there is an unreliable link between each pair of sufficiently close nodes. Gilbert’s question of multi-hop connectivity can then be rephrased as follows: can you find an infinite path in the resulting graph, if each time you need to traverse a new edge you flip a coin to decide if it is possible to do so?

It turns out that if we view results in terms of the average number of edges that can be traversed per node, the simplistic disc model used in continuum percolation, where edges in the connectivity graph correspond to reliable links and can always be traversed, is the “hardest” to percolate among many random connection models. In other words, the disc model doesn’t form an unbounded connected component until the density of discs is high enough that the expected number of connections (ENC) is above roughly 4.5,¹ which is a value higher than what required by many random connection models. This indicates that in practice real networks can exploit the presence of unreliable connections to achieve connectivity more easily, if they can maintain the average number of *functioning* connections. Numerical simulations show similar results in the case of

¹This is 4π times the more familiar percolation threshold density of .359... for unit discs.

anisotropic radiation patterns, where the disc appears the “most compact”, and hence the hardest to percolate, among all shapes of equivalent effective area, whether convex or not, and whether probabilistic or not.²

In the next section we present results on noisy communication channels. In Section 3 we present results on anisotropic radiation patterns. Section 4 concludes the paper.

II. PERCOLATION WITH NOISY LINKS

We assume all nodes to be distributed on the plane according to a bi-dimensional Poisson point process X , and we model imperfect links by considering a random connection model where each pair of points (x, y) of the point process is connected with probability $g(x - y)$, for some given function $g : \mathbb{R}^2 \rightarrow [0, 1]$.

We may, for example, pick a function $g(\cdot)$ such that the probability of existence of a link between a transmitter and a receiver decreases as the two points get farther away. For generality, however, we prefer to let g be an arbitrary function. Accordingly, we do not require $g(\cdot)$ to have bounded support, be spherically symmetric, nor monotonic. The only requirement (in order to avoid a trivial model) is that its *effective area* $e(g) = \int_{x \in \mathbb{R}^2} g(x)$ must be $0 < e(g) < \infty$. We call H the class of functions that satisfy this requirement.

The effective area, when multiplied by the density of points, gives the expected number of points connected to a point of the Poisson process, hence the two cases $e(g) = 0$ and $e(g) = \infty$ do not lead to any phase transition, since nodes always have respectively 0 or ∞ neighbors in those cases. (See Chapter 6 of [10] for details.) However, any function $g(\cdot)$ such that $0 < e(g) < \infty$, has a phase transition. Namely, there exists a critical value $\lambda_c(g)$ for the density of the Poisson process, such that

$$0 < \lambda_c(g) = \inf\{\lambda : \exists \text{ infinite connected component}\} < \infty.$$

Note that the standard continuum percolation model, where Poisson points are connected with probability one, if discs of radius r centered at each point overlap, is a random connection model with a connection function

$$g(x) = \begin{cases} 1 & \text{if } \|x\| \leq 2r \\ 0 & \text{if } \|x\| > 2r. \end{cases} \quad (1)$$

We are interested in how the percolation properties of the model change when we change the form of the connection function, while preserving its effective area. We start by considering “squishing and squashing” the connection function.

A. Squishing and squashing

Given a function $g \in H$ define g_p^{squash} by $g_p^{squash}(x) = p * g(\sqrt{p}x)$. This function, as illustrated in Figure 2, is a version of g in which probabilities are reduced by a factor of p but the function is stretched so as to maintain the original effective area. Therefore, the average number of connections of

²Note that aligned ellipses will percolate at the same density as discs of the same area, since an affine transformation of the plane can convert one shape into the other without affecting connectivity.

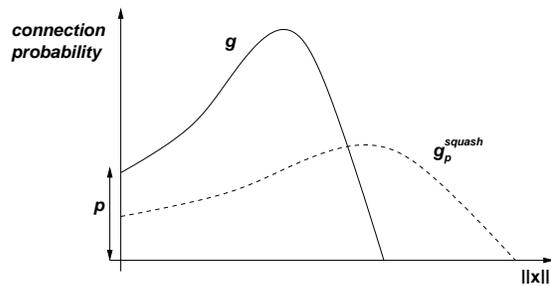


Fig. 2. **Squishing and squashing.** The function g is squished and squashed to give the function g_p^{squash} .

each point remains the same, but these connections have a wider range of lengths.

Note that considering the squashed version of a rectangular connection function such as the one in Eq. (1), corresponds to connecting Poisson points with probability p if their discs of radius \sqrt{pr} overlap.

We can now state the following theorem:

Theorem II.1: For all $g \in H$,

$$\lambda_c(g) \geq \lambda_c(g_p^{squash}).$$

Proof sketch of Theorem II.1 The proof is based on scaling and coupling arguments. The scaling involves finding a relation between the connection function $g(\sqrt{p}x)$ and $g_p^{squash}(x)$, and between $g(\sqrt{p}x)$ and $g(x)$. Indeed, one can obtain a graph based upon the connection function $g_p^{squash}(x)$ by removing edges independently and with probability $1 - p$ from the graph based upon the connection function $g(\sqrt{p}x)$. Similarly, one can obtain a graph based upon the connection function $g(x)$ by removing nodes independently and with probability $1 - p$ from the graph based upon the connection function $g(\sqrt{p}x)$. Then one needs to couple the two thinning processes in a way to obtain an infinite cluster after removing points only if there is an infinite cluster after removing edges. It turns out that such a (dynamic) coupling is not difficult to construct, and has been exploited in discrete percolation on graphs to prove the well known inequality $p_c(\text{site}) < p_c(\text{bond})$. We refer the reader to [4] for details. \square

The theorem above has a certain depth. Essentially, it states that unreliable links are at least as good at providing connectivity as reliable links, if the average number of connections per node is the same in each case. Another way of looking at this is that the longer links introduced by stretching the connection function are making up for the increased unreliability of the connections.

In some related work Penrose [12] has shown that (speaking roughly) as a connection function of effective area 1 gets more spread out, its critical density for percolation converges to 1. This can be seen as the limiting case of our Theorem II.1. Meester, Penrose, and Sarkar [11] proved a similar result as the dimension tends to infinity. Philosophically, the idea that some longer connections help to reach percolation at a lower density of points is also related to the small world networks described, for example, in the paper by Watts and Strogatz [16].

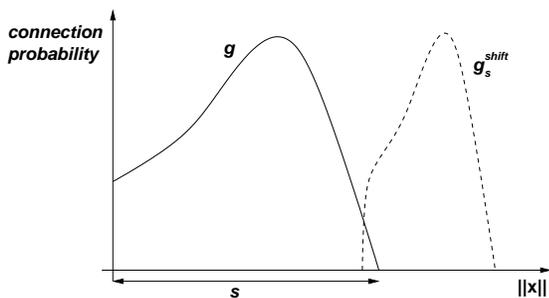


Fig. 3. **Shifting and squeezing** The function g is shifted and squeezed to give the function g_s^{shift} .

The dynamic coupling technique needed in the proof of the Theorem II.1 was first used by Grimmett and Stacey [4] to show the relationship between the critical values of site and bond percolation on graphs. Sites of the discrete model are the analog of Poisson points in our continuum model. Grimmett and Stacey [4] also show that bond percolation is strictly easier to percolate for some (broad) classes of graphs. Unfortunately, the kind of graphs we are considering here do not fall into any of those classes and we are left with the following conjecture:

Conjecture II.2: For all $g \in H$ and $p < 1$,

$$\lambda_c(g) > \lambda_c(g_p^{squash}).$$

We support Conjecture II.2 by numerical simulations, see Fig. 7.

B. Shifting and squeezing

Another transformation of g that we consider is $g_s^{shift}(x)$. Here we shift the function g outwards (so that a disc becomes an annulus, for example) by a distance s , but squeeze the function so that it has the same effective area. See Figure 3 for an example. Technically we define this through $g_s^{shift}(x) = g(h^{-1}(x))$ and

$$\int_s^{s+h(y)} r g_s^{shift}(r) dr = \int_0^y r g(r) dr.$$

Note that the shifting and squeezing transformation maintains on average the same number of bonds per node, but makes them all longer. This contrasts with the squishing and squashing transformation which gives more of a mixture of short and long edges. Is the stretching of the edges enough to help the percolation process, or do we need the mixture provided by squishing and squashing? We believe (supported by experimental evidence) that the longer edges introduced by the shifting and squeezing transformation are enough to help the percolation process. Accordingly, we make the following conjecture:

Conjecture II.3: For all $g \in H$ and $s > 0$,

$$\lambda_c(g) > \lambda_c(g_s^{shift}).$$

We support conjecture II.3 by numerical simulation, using a rectangular connection function of the type described by Eq. (1). Results in Figure 7 suggest that the shifting and squeezing conjecture is true (for this connection function), i.e., long

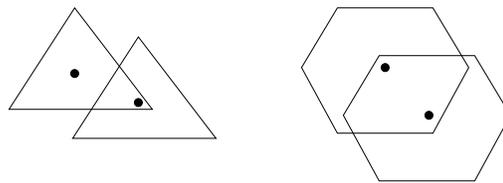


Fig. 4. **Central symmetry and asymmetry of shapes.** Left-hand side, centrally asymmetric shapes: the lower point is contained in the triangle around the higher point but not vice versa. Right-hand side, centrally symmetric shapes. The lower point is contained in the hexagon around the higher point and vice versa.

bonds are more useful for percolation than short ones at a given density of points. However, it must be noted that a scaling of the whole picture will make the bonds longer while changing the density of points. Naturally, this process does not change the connectivity, but is essentially different in that it changes the effective area.

At first sight, it may seem that Conjecture II.3 does not have an immediate practical application to wireless networks. However, we believe that proving it would provide insight on how to compare the percolation properties of arbitrary connection functions of the same effective area, which would be of great practical interest.

III. ANISOTROPIC RADIATION PATTERNS

We now discuss another extension to the connectivity model: non-rotationally symmetric transmission ranges. In practice, the communication range of a radio transmitter cannot be perfectly rotationally symmetric. The imperfection is due to the hardware characteristics of the transmitting antenna as well as the surrounding environment. It is therefore useful to understand how percolation properties are influenced by the shape of the transmitter footprint.

We consider centrally symmetric shapes that are not necessarily rotationally symmetric, i.e. shapes that are identical to themselves when rotated by 180 degrees. Accordingly, we let $B \subset H$ be the set of all connection functions of the form $b_C(x) = 1$ if x is inside some convex centrally symmetric shape C of area 1 and 0 otherwise. Let D be the disc of area 1 and let S be the square of area 1.

Centrally symmetric shapes allow us to consider only bi-directional links: a particularly nice property of centrally symmetric shapes is that two nodes overlap each other's centers if and only if the shapes scaled by a factor of two in each direction overlap each other. This means that if a node falls inside the shape of another node, then the reverse is true as well. This property does not hold for non-centrally symmetric shapes, see Fig 4.

Jonasson [9] has shown that if any convex shape of area 1 percolates at a certain density, then triangles (that are not centrally symmetric) of area 1 will also do so. Roy and Tanemura [13] strengthened this result to show a strict inequality between the critical density of triangle and that of any other given convex shape of the same area. Jonasson [9] has also shown that the convex shape with the highest critical density will be centrally symmetric. It is natural to ask which of the centrally symmetric

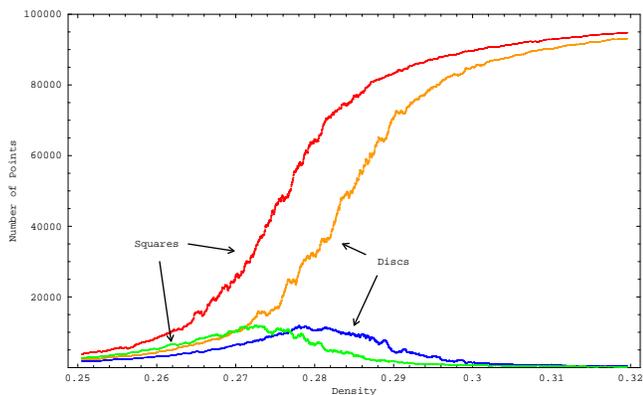


Fig. 5. **Size of larger clusters.** We report the size of the largest and second largest clusters of discs and squares of area one. The size of the largest cluster of discs tends to diverge at a smaller value of the density than for squares.

shapes percolate most and least easily. We believe the answers are the square and the disc.

Conjecture III.1: For all $b_C \in B$ we have

$$\lambda_c(b_S) \leq \lambda_c(b_C) \leq \lambda_c(b_D).$$

There is an important practical consequence that follows if Conjecture III.1 is true. If the disc is the centrally symmetric shape that percolates at the highest density, it follows from Jonasson [9] that this is also the highest percolating density shape over all the convex shapes. This means that percolation theory results on the existence of an unbounded connected cluster, derived in the standard model where overlapping discs are connected with probability one, are robust. If an ideal model, where transmission footprints are perfect discs, allows long-distance multi-hop communication, then long-distance multi-hop communication is also possible, under the same density conditions, in a less idealistic model where transmission footprints can have any convex shape with the same effective area as the disc.

We support Conjecture III.1 with computer simulations as shown in Figure 7. We have performed many computer simulations with different densities. We report the size of the largest and second largest clusters for discs and squares found in these simulations in Fig. 5. We ran overnight on a desktop computer a total of 7000 experiments for discs of unit area, using 100000 randomly placed discs for each experiment. The density of the discs in the experiments varied from a minimum of 0.25 to a maximum of 0.32, with a 0.00001 incremental step. For each density, the number of discs in the largest and second largest clusters were recorded. The same set of experiments were performed also on squares. Each data point shown in Fig. 5 corresponds to an average within a sliding window of 100 consecutive experiments.

Results show that the sizes of the largest two clusters of discs tend to diverge at a smaller value of the density than for squares. If we shift the plot obtained for squares by 3% (see Fig. 6), we find a striking matching of the curves. Accordingly, we conclude that squares seem to percolate at a value of the density 3% smaller than discs. This implies that the average number of

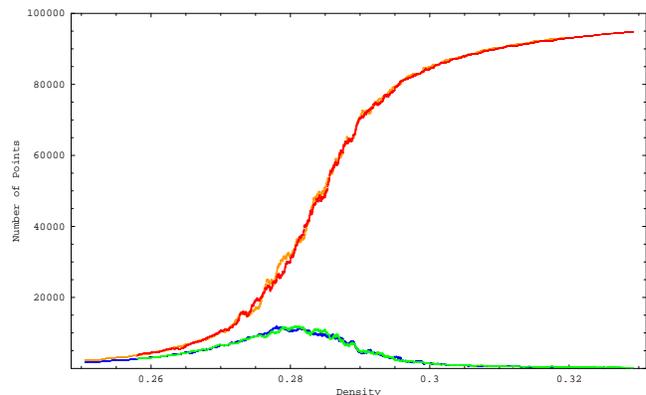


Fig. 6. **Shifting by 3%.** After shifting the curves of the size of the largest and second largest clusters of squares by 3% , we find a striking matching with the corresponding curves plotted for discs. This indicates that squares percolate at a value of the density approximately 3% smaller than discs.

connections needed for percolation (CNP) for squares is similarly reduced from the CNP for discs.

We can do similar experiments, using other shapes besides squares, to find the CNP for those shapes. Figure 7 shows the CNP for many different shapes and connection functions, computed in this way.

We conjecture that solid discs have the highest CNP of any shape, as suggested by the figure. The ascending shapes depicted on the left-hand side of the figure are discs with probabilistic connections to other points touching the disc, with probability varying linearly from 0.1 for the large light gray circle to 1.0 for the non-probabilistic disc at the top. Then the descending shapes are annuli whose inner radius varies linearly from 0.0 of the outer radius for the solid disc at the top (the very same shape as the last one in the previous sequence of ascending shapes) to 0.9 for the thin ring at the bottom. The “error bars” for each shape’s height are less than the distance between adjacent ticks shown on the vertical axis. The three shapes on the farther right are a solid disc (repeated for comparison with the hexagon), a hexagon, and a square. It would appear that $2n$ -gons have a CNP that increases with n , approaching the CNP for discs as n goes to infinity. The last shape on the right is an irregular lobed shape, inspired by the radiation pattern of a multidirectional antenna, and we see that its CNP is markedly lower than that of discs.

IV. CONCLUSIONS

We have studied the effects of unreliable communication and anisotropic radiation patterns on the connectivity of wireless ad-hoc networks. Following a percolation theory approach, we have shown that real networks can exploit the presence of unreliable connections and anisotropic radiation patterns, to achieve connectivity more easily, if they can maintain an average number of functioning connections. This result indicates that connectivity claims made in the literature using the simplistic assumption of reliable communication between nodes closer than a given distance, continue to hold in more realistic scenarios. Finally, our study leads to a number of open and challenging mathematical problems that need to be formally solved.

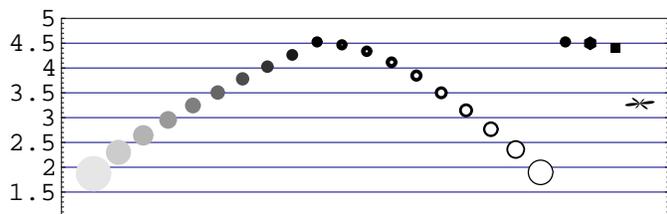


Fig. 7. **Percolation thresholds of various shapes.** Just as the curve for squares in Figure 6 lined up after being shifted by 3%, we can take the curve for any shape and see how much it needs to be expanded to achieve a good fit with the curve for discs. Since the percolation threshold for discs is known (experimentally), we assume that the percolation threshold for the other shape is at the same point in the graph, and so we divide the threshold for discs by the expansion factor needed for lining up the graphs to get an estimate of the percolation threshold for the other shape.

This process has been done for each of the shapes shown, and each shape, shown with effective area 1, is positioned so that the height of the center of the shape is at the CNP value for that shape. For example, the solid disc at the top is at a height of around 4.5, indicating that disc percolation occurs whenever each disc has over 4.5 neighbors on average. The bottom of the graph is a CNP of 1, which is the theoretical minimum, attained by very large, diffuse shapes. The “error bars” for each shape’s height are less than the distance between adjacent ticks shown on the vertical axis, except for the bottommost shapes, where the finiteness of the simulation size more easily interferes with the long range connectivities that are common in components formed by those shapes. The shapes on the left are discs with probabilistic connections to other points touching the disc, with probability varying linearly from 0.1 for the large light gray circle to 1.0 for the non-probabilistic disc at the top. Then the descending shapes are annuli whose inner radius varies linearly from 0.0 of the outer radius for the solid disc at the top (the very same shape as the last one in the previous sequence of ascending shapes) to 0.9 for the thin ring at the bottom. The next three shapes are a solid disc (repeated for comparison with the hexagon), a hexagon, and a square. It would appear that $2n$ -gons have a CNP that increases with n , approaching the CNP for discs as n goes to infinity. The last shape on the right is an irregular lobed shape, inspired by the radiation pattern of a multidirectional antenna, and we see that its CNP is markedly lower than that of discs.

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