Optimal Interleaving on Tori

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Abstract

This paper studies t-interleaving on two-dimensional tori, which is defined by the property that every connected subgraph of order t in the torus is labelled by t distinct integers. This is the first time that the t-interleaving problem is solved for graphs of modular structures. t-interleaving on tori has applications in distributed data storage and burst error correction, and is closely related to Lee metric codes. We say that a torus can be *perfectly t-interleaved* if its t-interleaving number — the minimum number of distinct integers needed to t-interleave the torus — meets the sphere-packing lower bound. We prove the necessary and sufficient conditions for tori that can be perfectly t-interleaved, and present efficient perfect t-interleaving constructions. The most important contribution of this paper is to prove that when a torus is large enough in both dimensions, its t-interleaving number is at most one more than the sphere-packing lower bound, and to present an optimal and efficient t-interleaving scheme for such tori. Then we prove bounds for the t-interleaving numbers of the remaining cases, completing a general characterization of the t-interleaving problem on 2-dimensional tori.

Index Terms

Burst, chromatic number, cluster, error-correcting code, multidimensional interleaving, t-interleaving, torus.

I. INTRODUCTION

Interleaving is an important technique used for error burst correction and network data storage. A most common example is the interleaving of n codewords in the form of '$1-2-3-\cdots-n-1-2-3-\cdots-n-\cdots\cdots$' for combatting one-dimensional error bursts in communication channels [24]. The concept of one-dimensional error burst was generalized to high dimensions by Blaum, Bruck and Vardy in [11], where an error burst of size t is a set of errors confined to a connected subgraph of order t in a multi-dimensional array. (The *order* of a graph is defined to be the number of vertices in that graph.) Accordingly, the concept of t-interleaving was defined in [11], which is a scheme to label the vertices of a multi-dimensional array with integers in such a way that every connected subgraph of order t is labelled by t distinct integers. t-interleaving schemes on two- and three-dimensional arrays were presented in [11], with applications in combatting error bursts in holographic storage systems and optical recording systems. Subsequent work on t-interleaving includes [30], where t-interleaving on circulant graphs with two offsets was studied, and [33], where a dual problem of t-interleaving on two-dimensional arrays was explored. The problem of two-dimensional interleaving with repetitions was introduced in [10] by Blaum, Bruck and Farrell, and was extensively studied in [13] by Etzion and Vardy. That problem is to interleave integers on a two-dimensional mesh (array or its variation) in such a way that in every connected subgraph of order t, each integer appears at most r times. Here t and r are given parameters, and the concept of interleaving with repetitions is a generalization of t-interleaving. More work on interleaving with repetitions includes [25] and [28]. Interleaving schemes on two-dimensional arrays achieving the Reiger bound were studied by Abdel-Ghaffar in [1], where error bursts of both rectangular shapes and arbitrary connected shapes were considered. More examples of interleaving for coping with error bursts include [4] and [9], where the error bursts are respectively of 'circular' types and rectangular shapes. As to interleaving schemes for network data storage, in [19], an algorithm was presented to interleave N integers on a tree whose edges have lengths, in such a way that for every point of the tree (including a vertex or a point on an edge), the smallest ball centered at the point that contains at least N integers contains all the N distinct integers. That algorithm is useful for distributed data storage in hierarchical networks that minimizes data retrieval delay. A related interleaving algorithm aiming at the graceful degradation of data-storage performance in faulty environments was presented in [20]. In [21], a scheme called *multi-cluster interleaving* was studied, which is a scheme to interleave integers on a path or a cycle such that every m disjoint intervals of length L in the path or cycle together contain at least K distinct integers, where $K > L$. Multi-cluster interleaving can be used for data storage on array-networks, ring-networks or disks where data are accessed through multiple access points.

In this paper, we study t-*interleaving* on two-dimensional tori. It is the first time that the t-interleaving problem on graphs of modular (wrapping-around) structures is solved. Torus is an important network structure for parallel and distributed systems [12], [26], [29], [31]. t-interleaving on tori has applications in both burst error correction and distributed data storage, in the same way as introduced in [11], [30], [33], [19] and [20]. (Specifically, for distributed data storage, a t-interleaving on a 2-dimensional torus ensures that for every vertex, the integers assigned within $\lfloor \frac{t-1}{2} \rfloor$ hops are all distinct.) t-interleaving on tori is also closely related to a research topic in coding theory called *Lee metric codes* [2], [3], [5], [6], [7], [8], [14], [15], [16], [17], [18], [22], [23], [27]. In a t-interleaved n-dimensional torus, every set of vertices labelled by the same integer is a Lee metric code of length n whose minimum distance is t; and the set of Lee metric codes corresponding to different integers partition the whole code space.

Below we present the definitions. t-interleaving was originally defined in [11] for arrays. We generalize its notion for general graphs straightforwardly.

Definition 1.1: Let G be a graph. We say that G is *interleaved* (or there is an *interleaving* on G) if every vertex of G is labelled by one integer. We say that G is t-*interleaved* (or there is a t-*interleaving* on G) if every connected subgraph of G of order t is labelled by exactly t distinct integers. $\square$

The classic vertex coloring problem is clearly also a t-interleaving problem, where $t = 2$. On the other hand, t-interleaving a graph G is the same as vertex-coloring the power graph G^t. Determining the chromatic number of a power graph is difficult in general. To the best of our knowledge, no result on the type of graphs we are interested in has appeared in the literature.

Definition 1.2: A two-dimensional $l_1 \times l_2$ torus is a graph containing $l_1 l_2$ vertices and $2 l_1 l_2$ edges. We denote its vertices by (i, j) for $0 \le i \le l_1 - 1$ and $0 \le j \le l_2 - 1$, in the way shown in the figure below:

$(0,0)$	$(0,1)$	$\cdots$	$(0,l_2-1)$
$(1,0)$	$(1,1)$	$\cdots$	$(1,l_2-1)$
$\vdots$	$\vdots$	$\ddots$	$\vdots$
$(l_1-1,0)$	$(l_1-1,1)$	$\cdots$	(l_1-1,l_2-1)

Each vertex (i, j) is incident to four edges, which connect it to its four neighbors $((i-1) \bmod l_1, j)$, $((i+1) \bmod l_1, j)$, $(i, (j-1) \bmod l_2)$ and $(i, (j+1) \bmod l_2)$. $\square$

Now we can define the problem of t-interleaving on tori.

Definition 1.3: Given a t-interleaved torus G, the number of distinct integers used to label the vertices of G is called the *degree* of this given t-interleaving scheme. The minimum degree of all the possible t-interleaving schemes for G is called the t-*interleaving number* of G. A t-interleaving on a torus whose degree equals the torus' t-interleaving number is called an *optimal* t-*interleaving*. $\square$

Example 1.1: The following 5×5 torus is 3-interleaved with degree 6.

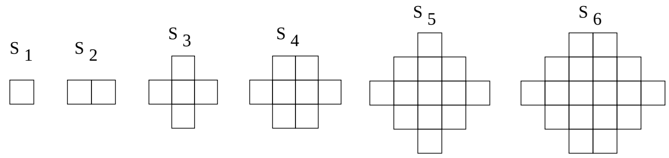


Fig. 1. Six examples of spheres.

0	3	1	4	2
1	4	2	0	3
2	0	3	1	5
3	1	5	2	0
4	2	0	3	1

If we replace the two integers ‘5’ with ‘4’, we will get a 3-interleaving with degree 5. Consider the vertex $(1, 1)$ and its four neighbors $(0, 1)$, $(2, 1)$, $(1, 0)$ and $(1, 2)$, and notice that any two of them are contained in a connected subgraph of order 3 — therefore any 3-interleaving scheme has to label those 5 vertices with 5 distinct integers. So the 3-interleaving number of this torus actually equals 5. $\square$

Our objective is to find optimal t-interleaving. To do that, it is important to obtain the t-interleaving numbers of tori. A universal lower bound of them, for tori that have at least t rows and t columns, can be obtained as follows. Figure 1 shows six subgraphs of a torus, which we call *spheres* S_1, S_2, $\cdots$, S_6, respectively. In general, for any $t \geq 3$, the sphere S_t is obtained by attaching to the sphere S_{t-2} all the vertices adjacent to it. Any two vertices in S_t are connected by a path of at most $t - 1$ edges, so a t-interleaving needs to label them with different integers. So the order of S_t, which we shall denote by $|S_t|$, sets a universal lower bound for the t-interleaving number. This argument was originally proposed in [11] for studying t-interleaving on arrays. A direct calculation tells us that $|S_t| = \frac{t^2+1}{2}$ when t is odd, and $|S_t| = \frac{t^2}{2}$ when t is even.

We define *perfect t-interleaving* to be a t-interleaving whose degree equals $|S_t|$, the universal lower bound, on a torus that has at least t rows and t columns. (A torus that does not satisfy that condition has only a very limited number of rows or columns; in this paper, we do not discuss the perfectness of interleaving for those tori.) We will show that a torus can be perfectly interleaved if and only if its sizes in both dimensions are multiples of a certain function of t. Then what about tori of other sizes? Our main result will show that when a torus is sufficiently large in both dimensions, its t-interleaving number exceeds the lower bound $|S_t|$ by at most one.

A more detailed description of our results is as follows:

- We prove that an $l_1 \times l_2$ torus can be perfectly t-interleaved if and only if the following condition is satisfied: when t is odd (respectively, even), both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ (respectively, t). We reveal the close relationship between perfect t-interleaving and perfect sphere packing, and present the *complete* set of perfect sphere packing constructions. Based on that, we obtain a set of efficient perfect t-interleaving constructions, which include the lattice interleaver scheme presented in [11] as a special case.
- We prove that for any torus that is sufficiently large in both dimensions, its t-interleaving number is either $|S_t|$ or $|S_t| + 1$ — that is, at most one more than the degree of perfect t-interleaving. More specifically, there exist integer pairs (θ_1, θ_2) such that whenever $l_1 \geq \theta_1$ and $l_2 \geq \theta_2$, the t-interleaving number of an $l_1 \times l_2$ torus is at most $|S_t| + 1$. Here θ_1 and θ_2 depend on t, and naturally, there is a tradeoff between them — if θ_1 takes a greater value, then the minimum value that θ_2 can take decreases or remains the same, and vice versa. We find a sequence of valid values for θ_1 and θ_2, which are shown in Theorem 10 and Theorem 11. We present optimal t-interleaving constructions for tori whose sizes exceed the found pairs (θ_1, θ_2). (And we comment that those constructions, as a general interleaving method, can also be used to optimally t-interleave tori of many other sizes.)
- We study upper bounds for t-interleaving numbers, and show that every $l_1 \times l_2$ torus’ t-interleaving number is $|S_t| + O(t^2)$.

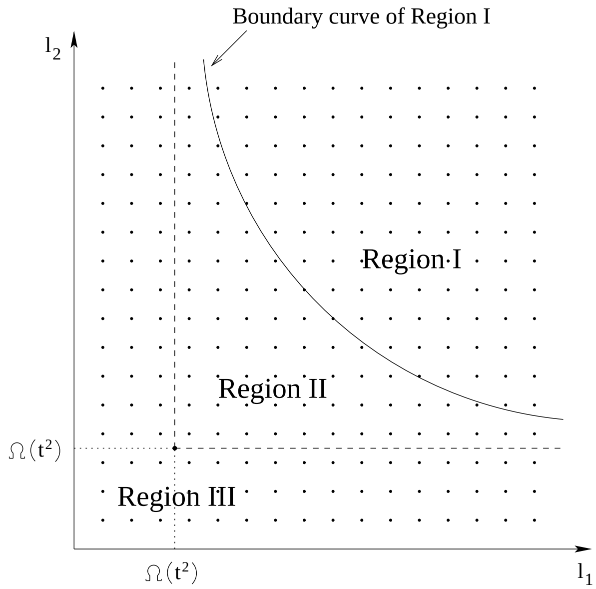


Fig. 2. A qualitative illustration of the t-interleaving numbers.

That upper bound is tight, even if $l_1 \to +\infty$ or $l_2 \to +\infty$. When both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.

The results can be illustrated qualitatively as Fig. 2. (The figure is not quantitative. The coordinates of points, such as the shape of the curve, are not exact.) Fig. 2 shows for any given t, how the $l_1 \times l_2$ tori can be divided into different classes based on their t-interleaving numbers.

The uniform lattice of dots in Fig. 2 are the sizes of all the tori that can be perfectly t-interleaved. The region labelled as *Region I* consists of all the integer pairs (θ_1, θ_2). The boundary curve of Region I is non-increasing, and symmetric with respect to the line $l_2 = l_1$. We know the exact t-interleaving number of every torus in this region — $|S_t|$ if it is one of the lattice dots, and $|S_t| + 1$ otherwise. The most important contribution of this paper is to prove the existence of Region I, and present the corresponding optimal interleaving constructions. Region II is the region where $l_1 = \Omega(t^2)$ and $l_2 = \Omega(t^2)$, in which the tori's t-interleaving numbers are upper-bounded by $|S_t| + O(t)$. Region III includes every torus, where the t-interleaving number is upper-bounded by $|S_t| + O(t^2)$. That upper bound for Region III is tight, even if l_1 or l_2 approaches $+\infty$. (So increasing a torus' size in only one dimension does not help reduce the t-interleaving number very effectively in general.)

The rest of the paper is organized as follows. In Section II, we show the necessary and sufficient conditions for tori that can be perfectly t-interleaved, and present perfect t-interleaving constructions based on perfect sphere packing. In Section III, we present a t-interleaving method, with which we can t-interleave large tori with a degree within one of the optimal. In Section IV, we improve upon the t-interleaving method shown in Section III, and present optimal t-interleaving constructions for tori whose sizes are large in both dimensions. As a parallel result, the existence of Region I is proved. In Section V, we prove some general bounds for the t-interleaving numbers. In Section VI, we conclude this paper.

II. Perfect t-Interleaving

In this section, we show the close relationship between *perfect t-interleaving* and *perfect sphere packing*, and use it to prove the necessary and sufficient condition for tori to have perfect t-interleaving. We present the complete set of perfect sphere packing constructions. Based on them, we derive efficient perfect t-interleaving constructions.

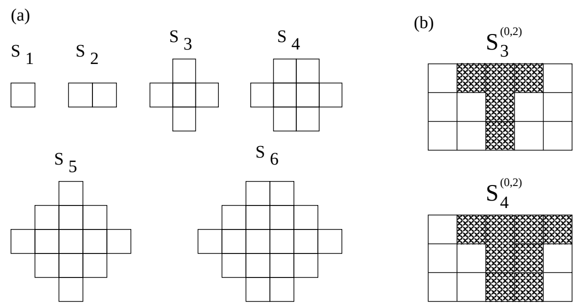


Fig. 3. Examples of the sphere S_t.

A. Perfect t-Interleaving and Sphere Packing

Definition 2.1: The *Lee distance* between two vertices in a torus is the number of edges in the shortest path connecting those two vertices. For two vertices in an $l_1 \times l_2$ torus G, (a_1, b_1) and (a_2, b_2), the *Lee distance* between them is denoted by $d((a_1, b_1), (a_2, b_2))$. (Therefore, $d((a_1, b_1), (a_2, b_2)) = \min\{(a_1 - a_2) \bmod l_1, (a_2 - a_1) \bmod l_1\} + \min\{(b_1 - b_2) \bmod l_2, (b_2 - b_1) \bmod l_2\}$.) Occasionally, in order to emphasize that the two vertices are in G, we also denote it by $d_G((a_1, b_1), (a_2, b_2))$. □

Clearly, an interleaving on a torus is a t-interleaving if and only if the Lee distance between any two vertices labelled by the same integer is at least t.

The following is a more detailed definition of spheres, compared to the one in the Introduction section.

Definition 2.2: Let G be an $l_1 \times l_2$ torus where $l_1 \geq 2\lfloor \frac{t-1}{2} \rfloor + 1$ and $l_2 \geq t$, and let (a, b) be a vertex in G. When t is odd, the *sphere centered at* (a, b), $S_t^{(a,b)}$, is defined to be the subgraph induced by all those vertices whose Lee distance to (a, b) is less than or equal to $\frac{t-1}{2}$. When t is even, the *sphere left-centered at* (a, b), $S_t^{(a,b)}$, is defined to be the subgraph induced by all those vertices whose Lee distance to either (a, b) or $(a, (b+1) \bmod l_2)$ is less than or equal to $\frac{t}{2} - 1$. (a, b) is called the *center* of $S_t^{(a,b)}$ if t is odd, or the *left-center* of $S_t^{(a,b)}$ if t is even. If we do not care where the sphere is centered or left-centered, then the sphere is simply denoted by S_t. The number of vertices in the sphere is denoted by $|S_t|$. □

Example 2.1: Fig. 3 (a) shows the spheres S_1 to S_6. Fig. 3 (b) shows two spheres, $S_3^{(0,2)}$ and $S_4^{(0,2)}$, in a 3×5 torus. □

For any $l_1 \times l_2$ torus where $l_1 \geq t$ and $l_2 \geq t$, its t-interleaving number is at least $|S_t|$. We call $|S_t|$ the *sphere packing lower bound*. The relationship between this bound and sphere packing will become clearer soon.

Definition 2.3: A torus G is said to have a *perfect packing of spheres* S_t if spheres S_t are packed in G in such a way that every vertex of G lies in exactly one of the spheres. □

Lemma 1: (1) Let t be odd. An interleaving on an $l_1 \times l_2$ torus (where $l_1 \geq t$ and $l_2 \geq t$) is a t-interleaving if and only if for any two vertices (a_1, b_1) and (a_2, b_2) that are labelled by the same integer, the two spheres centered at them, $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$, do not share any common vertex.

(2) Let t be even. An interleaving on an $l_1 \times l_2$ torus (where $l_1 \geq t - 1$ and $l_2 \geq t$) is a t-interleaving if and only if for any two vertices (a_1, b_1) and (a_2, b_2) that are labelled by the same integer, the two spheres with them as left-centers, $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$, do not share any common vertex and what's more, $b_1 \neq b_2$ or $(a_1 - a_2) \neq \pm(t-1) \bmod l_1$.

Proof: (1) Let t be odd. Both $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ are classic spheres with radius $\frac{t-1}{2}$. If the interleaving is a t-interleaving, then the Lee distance between (a_1, b_1) and (a_2, b_2) is at least $t = 2 \cdot \frac{t-1}{2} + 1$, so $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ must have no intersection. The converse is also true.

(2) Let t be even. We consider two cases — $b_1 = b_2$ and $b_1 \neq b_2$.

First consider the case '$b_1 = b_2$'. In this case, $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection if and only if $d((a_1,b_1),(a_2,b_2)) \geq 2\cdot(\frac{t}{2}-1)+1 = t-1$. And $d((a_1,b_1),(a_2,b_2)) = t-1$ if and only if $(a_1-a_2) \equiv \pm(t-1) \bmod l_1$. So the Lee distance between (a_1,b_1) and (a_2,b_2) is at least t if and only if $S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection and $(a_1-a_2) \not\equiv \pm(t-1) \bmod l_1$, which is the conclusion we want.

Now consider the case '$b_1 \neq b_2$'. In this case, the Lee distance between (a_1,b_1) and (a_2,b_2) is at least $t \Longleftrightarrow$ both the Lee distance between $(a_1,(b_1+1) \bmod l_2)$ and (a_2,b_2) and the Lee distance between $(a_2,(b_2+1) \bmod l_2)$ and (a_1,b_1) are at least $t-1 \Longleftrightarrow S_{t-1}^{(a_1,(b_1+1) \bmod l_2)}$ does not intersect $S_{t-1}^{(a_2,b_2)}$ and $S_{t-1}^{(a_2,(b_2+1) \bmod l_2)}$ does not intersect $S_{t-1}^{(a_1,b_1)} \Longleftrightarrow S_t^{(a_1,b_1)}$ and $S_t^{(a_2,b_2)}$ have no intersection. (Note that $S_t^{(a_1,b_1)}$ is the union of $S_{t-1}^{(a_1,b_1)}$ and $S_{t-1}^{(a_1,(b_1+1) \bmod l_2)}$, and $S_t^{(a_2,b_2)}$ is the union of $S_{t-1}^{(a_2,b_2)}$ and $S_{t-1}^{(a_2,(b_2+1) \bmod l_2)}$.) So we get the conclusion we want.

□

Theorem 1: For an $l_1 \times l_2$ torus where $l_1 \geq t$ and $l_2 \geq t$, if an interleaving on it is a perfect t-interleaving, then for every integer, the spheres S_t centered or left-centered at the vertices labelled by that integer form a perfect sphere packing in the torus. The converse is also true when $t \neq 2$.

Proof: Let's say that the torus is interleaved. We used I to denote the set of distinct integers used by the interleaving. For any integer $i \in I$, we use N_i to denote the number of vertices labelled by i.

Let's firstly prove one direction. Assume that the interleaving is a perfect t-interleaving. Then $|I| = |S_t|$. By Lemma 1, for any $i \in I$, the spheres S_t centered or left-centered at vertices labelled by i do not overlap. By counting the number of vertices in the torus and in each sphere S_t, we get $N_i \leq \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. Since $\sum_{i\in I} N_i = l_1 l_2$, we get $N_i = \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. So for any integer $i \in I$, the spheres S_t centered or left-centered at the vertices labelled by i form a perfect sphere packing in the torus.

Now let's prove the converse direction. Assume $t \neq 2$. And assume for every integer, the spheres S_t centered or left-centered at the vertices labelled by that integer form a perfect sphere packing in the torus. Then $N_i = \frac{l_1 l_2}{|S_t|}$ for any $i \in I$. Since $\sum_{i\in I} N_i = l_1 l_2$, we get $|I| = |S_t|$. What is left to prove is that the interleaving is a t-interleaving. By Lemma 1, the interleaving can fail to be a t-interleaving only if the following situation becomes true: "t is even, and there exist two vertices — (a_1,b_1) and (a_2,b_2) — labelled by the same integer such that $b_1 = b_2$ and $a_1 - a_2 \equiv t-1 \bmod l_1$." We will show that such a situation cannot happen.

Assume that situation happens. Then it is straightforward to verify that the following four vertices — $(a_1 - (\frac{t}{2}-1) \bmod l_1, b_1)$, $(a_2 + (\frac{t}{2}-1) \bmod l_1, b_1)$, $(a_1 - (\frac{t}{2}-2) \bmod l_1, b_1 - 1 \bmod l_2)$, $(a_2 + (\frac{t}{2}-2) \bmod l_1, b_1 - 1 \bmod l_2)$ — are contained in either $S_t^{(a_1,b_1)}$ or $S_t^{(a_2,b_2)}$, while the following two vertices — $(a_1 - (\frac{t}{2}-1) \bmod l_1, b_1 - 1 \bmod l_2)$ and $(a_2 + (\frac{t}{2}-1) \bmod l_1, b_1 - 1 \bmod l_2)$ — are neither contained in $S_t^{(a_1,b_1)}$ nor in $S_t^{(a_2,b_2)}$. The two vertices, $(a_1 - (\frac{t}{2}-1) \bmod l_1, b_1 - 1 \bmod l_2)$ and $(a_2 + (\frac{t}{2}-1) \bmod l_1, b_1 - 1 \bmod l_2)$, cannot both be contained in spheres S_t that are left-centered at vertices labelled by the same integer which labels (a_1,b_1) and (a_2,b_2), because they are vertically adjacent, and the vertices directly above them, below them and to the right of them are all contained in two spheres that do not contain them. (To see that, observe the shape of a sphere.) That contradicts that fact that all the spheres S_t left-centered at the vertices labelled by the integer which labels (a_1,b_1) form a perfect sphere packing in the torus. So the assumed situation cannot happen. By summarizing the above results, we see that the interleaving must be a perfect t-interleaving.

□

Theorem 2: For an $l_1 \times l_2$ torus where $l_1 \geq t$ and $l_2 \geq t$, if it can be perfectly t-interleaved, then the spheres S_t can be perfectly packed in it. The converse is also true when $t \neq 2$.

Proof: Let G be an $l_1 \times l_2$ torus. For any t, Theorem 1 has shown that if G can be perfectly t-interleaved, then the spheres S_t can be perfectly packed in it. Now we prove the other direction. Assume $t \neq 2$, and the spheres S_t can be perfectly packed in G. Let (x_1,y_1), (x_2,y_2), $\cdots$, (x_n,y_n) be a set of vertices such that the spheres S_t centered or left-centered at them form a perfect packing in G. The proof of Theorem 1 has essentially showed that for any i and j ($i \neq j$), the Lee distance between

(x_i, y_i) and (x_j, y_j) is at least t. Now we can interleave G is this way: label each sphere S_t with $|S_t|$ distinct integers such that every integer is used exactly once in every sphere, and make all the spheres to be labelled in the same way (namely, all the spheres have the same 'interleaving pattern'). Clearly, for any two integers a and b, the two sets of vertices respectively labelled by a and b are cosets of each other in the torus — therefore the Lee distance between any two vertices labelled by the same integer is at least t. So G has a perfect t-interleaving.

□

B. Perfect t-Interleaving and Its Construction

The following lemma is an important property of perfect sphere packing. It will help us derive the necessary and sufficient condition for perfect t-interleaving.

Lemma 2: Let t be even and $t \geq 4$. When spheres S_t are perfectly packed in an $l_1 \times l_2$ torus, there exists an integer $a \in \{+1, -1\}$, such that if there is a sphere left-centered at the vertex (x, y), then there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y - a \cdot \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y + a \cdot \frac{t}{2}) \bmod l_2)$.

Proof: Assume spheres S_t are perfectly packed in an $l_1 \times l_2$ torus, where $t \geq 4$ and t is even. Firstly, we will show that $l_1 \geq t$. Since t is even, a sphere S_t spans $t - 1$ rows. So $l_1 \geq t - 1$. Now we show why $l_1 \neq t - 1$. Fig. 4 (a) shows two examples — the first example shows a sphere S_4 in a torus of 3 rows, and the second example shows a sphere S_6 in a torus of 5 rows. (The vertices in the two spheres are indicated by relatively large black dots in the figure.) Considering the shapes of the spheres, we can easily see that the two adjacent vertices in each dashed circle cannot be both contained in non-overlapping spheres. Such a phenomenon always happens when $l_1 = t - 1$. Since here spheres S_t are perfectly packed in the torus, we get $l_1 \geq t$.

Clearly, one of the following two cases must be true:

- Case 1: whenever there is a sphere left-centered at a vertex (x, y), there are four spheres respectively left-centered at the four vertices $((x - \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$, $((x - \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$, $((x + \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$.
- Case 2: there exists a sphere left-centered at a vertex (x_0, y_0), such that there is no sphere left-centered at at least one of the following four vertices — $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 - \frac{t}{2}) \bmod l_2)$, $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$, $((x_0 + \frac{t}{2}) \bmod l_1, (y_0 - \frac{t}{2}) \bmod l_2)$ and $((x_0 + \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$.

If Case 1 is true, then the conclusion of this lemma obviously holds. From now on, let us assume that Case 2 is true. WLOG (without loss of generality), we assume that there is one sphere left-centered at (x_0, y_0), but there is no sphere left-centered at $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + \frac{t}{2}) \bmod l_2)$. (All the other possible instances can be proved with the same method.)

Since $l_1 \geq t$, the vertex $((x_0 - \frac{t}{2}) \bmod l_1, (y_0 + 1) \bmod l_2)$ — which we shall call 'vertex A' — is not contained in the sphere left-centered at (x_0, y_0). (An example is shown in Fig. 4 (b), where the sphere in consideration is an S_t with $t = 8$, whose left-center (x_0, y_0) is labelled by 'C'. The vertex A is labelled by 'A'.) The vertex A is contained in one of the perfectly packed spheres, which we shall call 'sphere B'. The relative position of vertex A in sphere B can only be one of the following two possibilities:

- Possibility 1: the vertex A is the right-most vertex in the bottom row of the sphere B. (See Fig. 5 (a).)
- Possibility 2: the vertex A is in the down-left diagonal of the border of the sphere B, but it is not the left-most vertex of the sphere B. (See Fig. 5 (b), (c) and (d).)

Possibility 1, however, can be easily found to be impossible, since otherwise the neighboring vertex to the right of vertex A and the vertex below it cannot both be contained in non-overlapping spheres. (See the two vertices in the dashed circle in Fig. 5 (a).) So only possibility 2 is true. In the following proof we use the example of $t = 8$ for illustration, and assume that the relative position of the sphere B is as shown in Fig. 5 (b). We comment that when t takes other values or when the sphere B takes other relative positions, the following argument still holds, which will be easy to see.

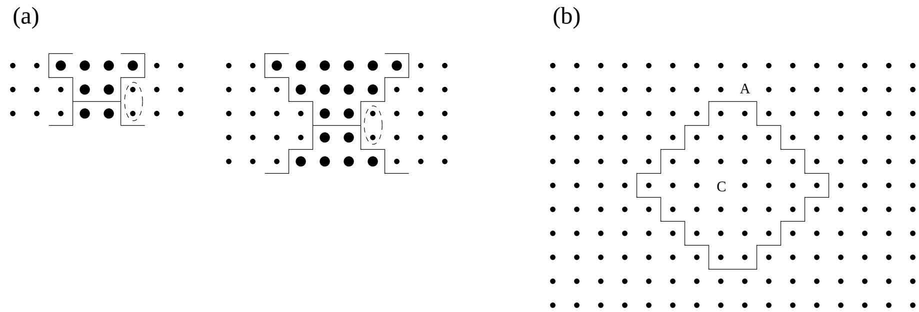


Fig. 4. A sphere in a torus.

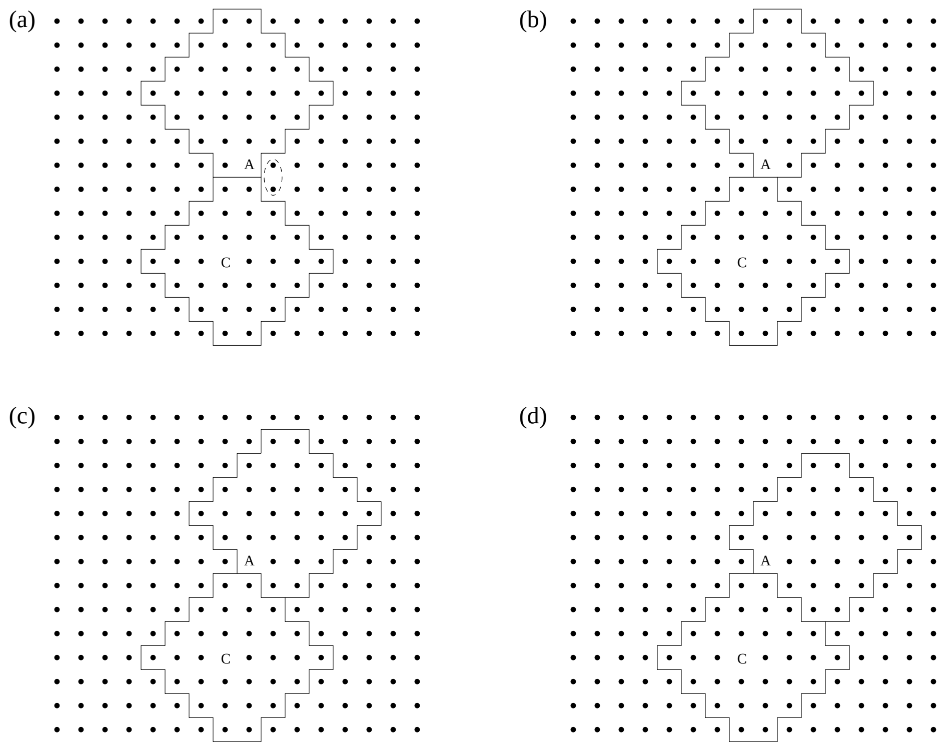


Fig. 5. Relative positions of spheres and vertices.

Let the sphere left-centered at (x_0, y_0) be the sphere denoted by 'L_1' in Fig. 6, and let sphere B be the sphere now denoted by 'R_1' in Fig. 6. We immediately see that the vertex denoted by 'E' must be the right-most vertex of a sphere, so the sphere containing the vertex 'E' must be the sphere denoted by 'L_2'. Then we immediately see that the vertex denoted by 'F' must be the right-most vertex in the bottom row of a sphere, so the sphere containing the vertex 'F' must be the sphere denoted by 'R_2'. With the same method we can fix the positions of a series of spheres L_1, L_2, L_3, L_4, $\cdots$ and a series of spheres R_1, R_2, R_3, R_4, $\cdots$. Since the torus is finite, we will get a series of spheres L_1, L_2, L_3, L_4, $\cdots$, L_n such that the relative position of L_n to L_1 is the same as the relative position of L_1 to L_2 (see Fig. 6 for an illustration) — so such a series of spheres form a 'cycle' in the torus. Since the spheres are perfectly packed in the torus, no two spheres in this 'cycle' overlap. Similarly, the spheres R_1, R_2, $\cdots$, R_n also form a 'cycle' in the torus. (Note that we do not make any assumption about whether these two 'cycles' overlap or not.)

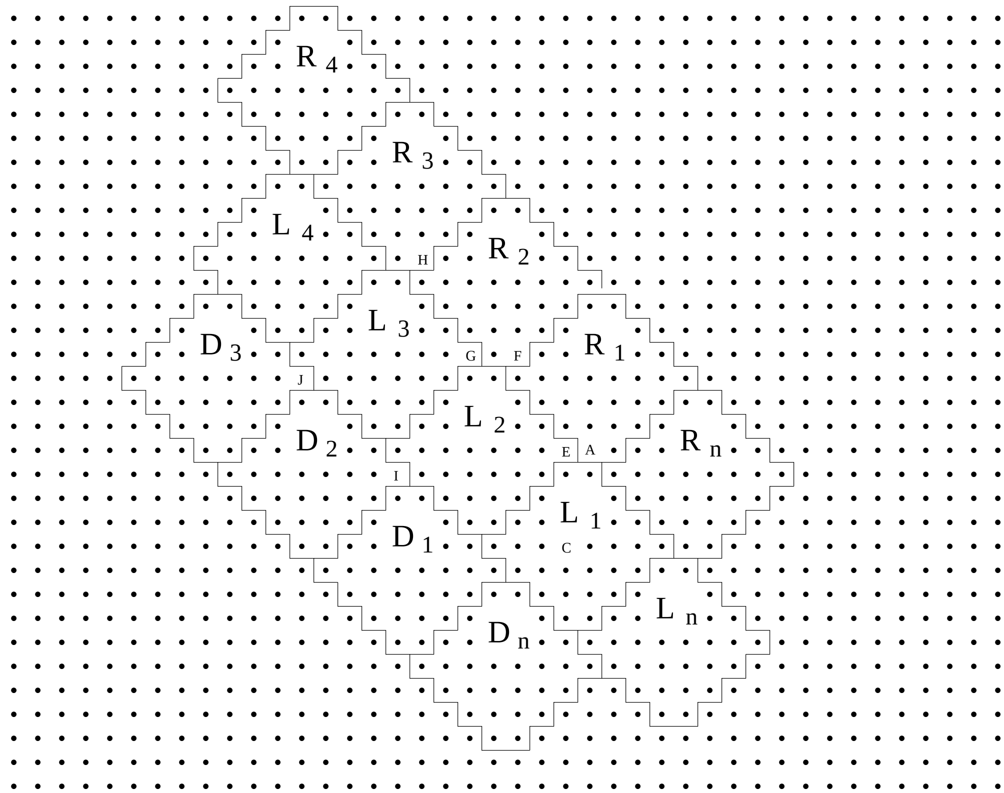


Fig. 6. The packing of spheres in a torus.

If those two 'cycles' contain all the spheres in the torus, then we are already very close to the end of this proof. If those two 'cycles' do not contain all the spheres in the torus, then there must be some spheres outside the two 'cycles' that are directly attached to the down-left side of the 'cycle' formed by L_1, L_2, $\cdots$, L_n. (Consider the very regular way the 'cycle' is formed, and the resulting shape of the 'cycle' which is invariant to horizontal and vertical shifts.) Let D_1 be a sphere directly attached to the 'cycle' formed by L_1, L_2, $\cdots$, L_n, as shown in Fig. 6. (Note that we do not care about the exact position of D_1, as long as it is directly attached to the down-left side of the 'cycle'.) Then the vertex 'I' immediately determines that the sphere containing it must be 'D_2'; similarly the vertex 'J' determines the position of the sphere 'D_3'; and so on $\cdots\cdots$ So we will get a series of spheres D_1, D_2, D_3, $\cdots$, D_n which will again form a 'cycle'. (It is easy to see that this 'cycle' does not overlap the previous two 'cycles'.) With the same method as above, we will find more and more 'cycles', until they together contain all the spheres in the torus.

We can easily see that in each of the 'cycles' here, if there is a sphere left-centered at a vertex (x, y), then there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$. When other instances of Case 2 are true (see the definition of 'Case 2' in previous text), it can be shown in the same way that whenever there is a sphere left-centered at a vertex (x, y), there are two spheres respectively left-centered at $((x - \frac{t}{2}) \bmod l_1, (y + \frac{t}{2}) \bmod l_2)$ and $((x + \frac{t}{2}) \bmod l_1, (y - \frac{t}{2}) \bmod l_2)$. By summarizing the above conclusions, we see that this lemma is proved.

$\square$

Definition 2.5: Let t be an even positive integer, let a be either $+1$ or -1, and let G be an $l_1 \times l_2$ torus. Let (x, y) be an arbitrary vertex in G. We define "the *cycle* containing (x, y) (corresponding to the parameter a)" to be the set of spheres S_t that are respectively left-centered at the vertices (x, y), $((x + \frac{t}{2}) \bmod l_1, (y + a \cdot \frac{t}{2}) \bmod l_2)$, $((x + 2 \cdot \frac{t}{2}) \bmod l_1, (y + 2a \cdot \frac{t}{2}) \bmod l_2)$, $((x + 3 \cdot \frac{t}{2}) \bmod l_1, (y + 3a \cdot \frac{t}{2}) \bmod l_2)$, $\cdots\cdots$ $\square$

The proof of the following lemma is omitted due to its simplicity.

Lemma 3: Let t be an even positive integer, let a be either $+1$ or -1, and let G be an $l_1 \times l_2$ torus. For any vertex (x, y) in G, the *cycle* containing it (corresponding to the parameter a) consists of $\frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}}$ distinct spheres S_t.

The following theorem shows the necessary and sufficient condition for tori that can be perfectly t-interleaved.

Theorem 3: Let G be an $l_1 \times l_2$ torus where $l_1 \geq t$ and $l_2 \geq t$. If t is odd, then G can be perfectly t-interleaved if and only if both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$. If t is even, then G can be perfectly t-interleaved if and only if both l_1 and l_2 are multiples of t.

Proof: We consider the following three cases one by one:

- Case 1: $t = 2$.
- Case 2: t is even but $t \neq 2$.
- Case 3: t is odd.

Case 1: $t = 2$. In this case, 2-interleaving is equivalent to vertex coloring, so the 2-interleaving number of G equals G's chromatic number $\chi(G)$. Let R_1 and R_2 be two rings which respectively have l_1 and l_2 vertices. Then G is the Cartesian product of those two rings, namely, $G = R_1 \otimes R_2$. It is well known [32] that for any two graphs H_1 and H_2, $\chi(H_1 \otimes H_2) = \max\{\chi(H_1), \chi(H_2)\}$. Since $l_1 \geq t = 2$ (respectively, $l_2 \geq t = 2$), we get that $\chi(R_1) \geq 2$ (respectively, $\chi(R_2) \geq 2$); and $\chi(R_1) = 2$ (respectively, $\chi(R_2) = 2$) if and only if l_1 (respectively, l_2) is a multiple of 2. So $\chi(G) = 2$ if and only if both l_1 and l_2 are multiples of 2. Since $|S_2| = 2$, we get the conclusion in this lemma.

Case 2: t is even but $t \neq 2$. Firstly, we prove one direction. Assume G can be perfectly t-interleaved. We will show that both l_1 and l_2 are multiples of t. Let i be an integer used by a perfect t-interleaving on G. Then by Theorem 1, the spheres S_t left-centered at the vertices labelled by i form a perfect sphere packing in G. By Lemma 2, there exists an integer $a \in \{+1, -1\}$ such that for any *cycle* containing a vertex labelled by i (corresponding to the parameter a), the spheres S_t in the *cycle* are all left-centered at vertices labelled by i — and therefore they do not overlap. By Lemma 3, the *cycle* containing a vertex labelled by i consists of $\frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}}$ distinct spheres S_t. So such a *cycle* consists of $\frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}} \cdot |S_t| = \frac{lcm(l_1, l_2, \frac{t}{2})}{\frac{t}{2}} \cdot \frac{t^2}{2} = lcm(l_1, l_2, \frac{t}{2}) \cdot t$ vertices. Let (x_1, y_1) and (x_2, y_2) be any two vertices labelled by i. We can see that for the *cycle* containing (x_1, y_1) and the *cycle* containing (x_2, y_2), they either do not overlap, or they are the same *cycle*. Therefore, the vertices in G can be partitioned into several such *cycles* — so $l_1 \cdot l_2$ is a multiple of $lcm(l_1, l_2, \frac{t}{2}) \cdot t$. Since $lcm(l_1, l_2, \frac{t}{2})$ is a multiple of l_1, l_2 must be a multiple of t. Similarly, l_1 must be a multiple of t, too. So if G can be perfectly t-interleaved, then both l_1 and l_2 are multiples of t.

Now we prove the other direction. Assume both l_1 and l_2 are multiples of t. Let W be such a set of vertices in G: $W = \{(x, y) | x \equiv 0 \bmod \frac{t}{2}, y \equiv 0 \bmod \frac{t}{2}, x + y \equiv 0 \bmod t\}$. It is easy to verify that the Lee distance between any two vertices in W is at least t. Now for $i = 0, 1, \cdots, \frac{t}{2} - 1$ and for $j = 0, 1, \cdots, t - 1$, define $W^{i,j}$ to be $W^{i,j} = \{((x + i) \bmod l_1, (y + j) \bmod l_2) | (x, y) \in W\}$. Clearly those $\frac{t}{2} \cdot t = |S_t|$ sets — $W^{0,0}$, $W^{0,1}$, $\cdots$, $W^{\frac{t}{2}-1, t-1}$ — is a partition of the vertices in G. For each $W^{i,j}$, we label the vertices in it with one distinct integer. Clearly such an interleaving is a perfect t-interleaving. So if both l_1 and l_2 are multiples of t, then G can be perfectly t-interleaved.

Case 3: t is odd. Firstly, we prove one direction. Assume both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$. Golomb and Welch have shown in [15] that an $\frac{t^2+1}{2} \times \frac{t^2+1}{2}$ torus can be perfectly packed by the spheres S_t for odd t. Therefore, G can also be perfectly packed by S_t because a torus has a toroidal topology and G can be 'folded' into an $\frac{t^2+1}{2} \times \frac{t^2+1}{2}$ torus. Let C be a set of vertices in G such that the spheres S_t centered at the vertices in C form a perfect sphere packing. Then the Lee distance between any two vertices in C is at least t. We call a set of vertices D a *coset of* C when the following condition is satisfied: "there exist integers a and b such that a vertex $(x, y) \in C$ if and only if $((x + a) \bmod l_1, (y + b) \bmod l_2) \in D$." C has $|S_t|$ different cosets in total (including C itself), and those cosets partition the vertices of G. For each coset, we label its vertices with one distinct integer, and we get a perfect t-interleaving. So if both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$, then G can be perfectly t-interleaved.

Now we prove the other direction. Assume G can be perfectly t-interleaved. Let i be an integer used by a perfect t-interleaving on G. Then by Theorem 1, the spheres S_t centered at the vertices labelled by i form a perfect sphere packing in G. Golomb and Welch presented in [15] a way to perfectly pack spheres S_t in a torus when t is odd, which can be described as "either of the following two conditions is true: (1) whenever there is a sphere S_t centered at a vertex (x, y), there are two spheres respectively centered at $((x + \frac{t+1}{2}) \bmod l_1, (y + \frac{t-1}{2}) \bmod l_2)$ and $((x - \frac{t-1}{2}) \bmod l_1, (y + \frac{t+1}{2}) \bmod l_2)$; (2) whenever there is a sphere S_t centered at a vertex (x, y), there are two spheres respectively centered at $((x + \frac{t-1}{2}) \bmod l_1, (y + \frac{t+1}{2}) \bmod l_2)$ and $((x - \frac{t+1}{2}) \bmod l_1, (y + \frac{t-1}{2}) \bmod l_2)$". It is well known that that way of packing is in fact the only way to perfectly pack S_t for odd t, whose feasibility requires both l_1 and l_2 to be multiples of $\frac{t^2+1}{2}$. So if G can be perfectly t-interleaved, then both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$.

□

Below we present the complete set of perfect sphere packing constructions. But firstly let's explain a few concepts. Let G be an $l_1 \times l_2$ torus that is perfectly packed by spheres S_t — there are $\frac{l_1 l_2}{|S_t|}$ such spheres. Define e as $e = \frac{l_1 l_2}{|S_t|}$, and let's say those spheres are centered (or left-centered) at the vertices (x_1, y_1), (x_2, y_2), $\cdots$, (x_e, y_e). By *vertically (respectively, horizontally) shifting the spheres in G*, we mean to select some integer s, and get a new set of perfectly packed spheres that are centered (or left-centered) at $(x_1 + s \bmod l_1, y_1)$, $(x_2 + s \bmod l_1, y_2)$, $\cdots$, $(x_e + s \bmod l_1, y_e)$ (respectively, at $(x_1, y_1 + s \bmod l_2)$, $(x_2, y_2 + s \bmod l_2)$, $\cdots$, $(x_e, y_e + s \bmod l_2)$). By *vertically reversing the spheres in G*, we mean to get a new set of perfectly packed spheres that are centered (or left-centered) at $(-x_1 \bmod l_1, y_1)$, $(-x_2 \bmod l_1, y_2)$, $\cdots$, $(-x_e \bmod l_1, y_e)$. After such a 'shift' or 'reverse' operation, technically speaking, the way the spheres are perfectly packed in G are changed — however, the 'pattern of the sphere packing' essentially remains the same.

Construction 2.1: The complete set of perfect sphere packing constructions

Input: A positive integer t. An $l_1 \times l_2$ torus G, where (1) both l_1 and l_2 are multiples of t if t is even and $t \neq 2$, (2) l_2 is even if $t = 2$, and (3) both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ if t is odd.

Output: A perfect packing of the spheres S_t in G.

Construction:

1. If t is even and $t \neq 2$, then do the following:

- Let A_1, A_2, $\cdots$, $A_{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1}$ be $gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1$ integers, where A_i can be any integer in the set $\{0, 1, \cdots, \frac{t}{2} - 1\}$ for $i = 1, 2, \cdots, gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1$.
- Find the $gcd(\frac{l_1}{t}, \frac{l_2}{t})$ *cycles* in G (corresponding to the parameter 1) respectively containing the vertex $(0, 0)$, $(\sum_{i=1}^{1} A_i, \sum_{i=1}^{1}(t + A_i))$, $(\sum_{i=1}^{2} A_i, \sum_{i=1}^{2}(t + A_i))$, $\cdots$, $(\sum_{i=1}^{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1} A_i, \sum_{i=1}^{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1}(t + A_i))$. The spheres S_t in those $gcd(\frac{l_1}{t}, \frac{l_2}{t})$ *cycles* form a perfect sphere packing in the torus.

2. If $t = 2$, the do the following:

- The $l_1 \times l_2$ torus G has l_1 rows, each of which can be seen as a ring of l_2 vertices. When $t = 2$, the sphere S_t simply consists of two horizontally adjacent vertices. Split each row of G into $\frac{l_2}{2}$ spheres in any way. The resulting $\frac{l_1 l_2}{2}$ spheres form a perfect sphere packing in the torus.

3. If t is odd, then do the following:

- Find such a set of $\frac{l_1 l_2}{|S_t|}$ spheres S_t: each of the spheres is centered at a vertex $(i(m+1) + j \cdot (-m) \bmod l_1, i \cdot m + j(m+1) \bmod l_2)$ for some integers i and j. Those spheres form a perfect sphere packing in the torus.

4. Horizontally shift, vertically shift, and/or vertically reverse the spheres in G in any way.

□

Theorem 4: Construction 2.1 is the *complete* set of perfect sphere packing constructions.

Proof: We consider the following three cases. For each case, we need to prove two things: firstly, the 'Input' part of Construction 2.1 sets the necessary and sufficient condition for a torus to have perfect sphere packing; secondly, the 'Construction' part of Construction 2.1 generates perfect sphere packing correctly, and every perfect sphere packing that exists is a possible output of it.

Case 1: t is even and $t \neq 2$. In this case, since a sphere S_t occupies $t-1$ rows and t columns, for the $l_1 \times l_2$ torus G to have perfect sphere packing, it must be that $l_1 \geq t-1$ and $l_2 \geq t$. We can show that $l_1 \neq t-1$ in the following way — assume $l_1 = t-1$ and spheres S_t are perfectly packed in G; say a sphere S_t is left-centered at (x, y) in G; then the two vertices, $(x - (\frac{t}{2} - 1) \bmod l_1, y - 1 \bmod l_2)$ and $(x + (\frac{t}{2} - 1) \bmod l_1, y - 1 \bmod l_2)$, cannot both be contained in spheres (see the proof of Theorem 1 for a very similar argument), and that contradicts the statement that spheres are perfectly packed in G. Therefore, if G can be perfectly packed by spheres, $l_1 \geq t$ and $l_2 \geq t$. Then, from Theorem 2 and Theorem 3, we see that G can be perfectly packed by spheres if and only if both l_1 and l_2 are multiples of t. So the 'Input' part of Construction 2.1 correctly sets of the necessary and sufficient condition for a torus to have perfect sphere packing.

Lemma 2 and its proof have shown that when spheres are perfectly packed in a torus, those spheres can be partitioned into *cycles*. By observing the shape of the border of a *cycle*, we see that two adjacent *cycles* can freely 'slide' along each other's border — and there are $\frac{t}{2}$ possible relative positions between two adjacent *cycles*. In Construction 2.1, the $\frac{t}{2}$ possible relative positions are determined by A_i, a variable that can take $\frac{t}{2}$ possible values. Now it is easy to see that Step 1 of Construction 2.1 provides a perfect sphere packing (which takes one of many possible forms, depending on the value of the 'A_i's), and its Step 4 changes the positions of the spheres to furthermore cover all the possible cases of perfect sphere packing.

(2) Case 2: $t = 2$. We skip the proof for this case due to its simplicity.

(3) Case 3: t is odd. In this case, Construction 2.1 re-produces the sphere-packing method presented in [15], which is commonly known as the unique way to pack spheres for odd t (see the final paragraph of the proof of Theorem 3 for a more detailed introduction).

□

Now we present perfect t-interleaving constructions that are based on perfect sphere packing.

Construction 2.2: Perfect t-interleaving constructions

Input: A positive integer t. An $l_1 \times l_2$ torus G, where both l_1 and l_2 are multiples of t if t is even, and both l_1 and l_2 are multiples of $\frac{t^2+1}{2}$ if t is odd.

Output: A perfect t-interleaving on G.

Construction:

(1) If $t \neq 2$, then do the following:

- Use Construction 2.1 to get a perfect sphere packing in G. Label each of those spheres with $|S_t|$ distinct integers, in such a way that all the spheres have the same interleaving pattern, and every integer is used exactly once in each sphere.

(2) If $t = 2$, then do the following:

- For every vertex (i, j) of G, if $i + j$ is even, label it with the integer '0', otherwise label it with the integer '1'.

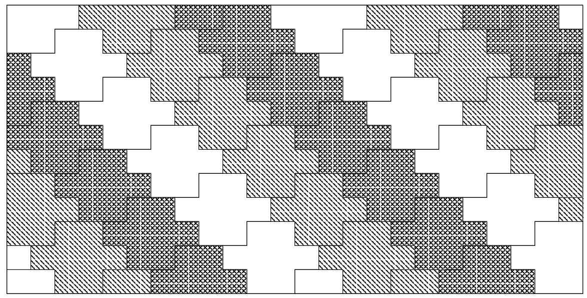
□

The following example illustrates how to use Construction 2.1 to obtain perfect sphere packing, and how to use Construction 2.2 to obtain perfect t-interleaving.

Example 2.2: Let $t = 4$, and let G be an 12×24 torus. Firstly, we use Construction 2.1 to find a perfect sphere packing in G. Since t is even, the Step 1 of Construction 2.1 is executed. We choose A_1, A_2, $\cdots$, $A_{gcd(\frac{l_1}{t}, \frac{l_2}{t})-1}$ to be $A_1 = 0$, $A_2 = 1$. (Note that here $gcd(\frac{l_1}{t}, \frac{l_2}{t}) - 1 = 2$.) Then the $gcd(\frac{l_1}{t}, \frac{l_2}{t}) = 3$ *cycles* in G are as shown in Fig. 7 (a), which are three sets of spheres S_t respectively of three different background shades. The spheres in those 3 *cycles* form a perfect packing in G.

Next, we use Construction 2.2 to perfectly t-interleave G. Let the perfect sphere packing remain as it is; and label all the spheres with the same interleaving pattern, using $|S_t| = 8$ distinct integers. The resulting perfect t-interleaving on G is shown in Fig. 7 (b). □

(a) G



(b) G

4	5	6	3	4	5	6	7	8	1	2	3	4	5	6	3	4	5	6	7	8	1	2	3
7	8	1	2	7	8	1	2	3	4	5	6	7	8	1	2	7	8	1	2	3	4	5	6
2	3	4	5	6	3	4	5	6	7	8	1	2	3	4	5	6	3	4	5	6	7	8	1
5	6	7	8	1	2	7	8	1	2	3	4	5	6	7	8	1	2	7	8	1	2	3	4
8	1	2	3	4	5	6	3	4	5	6	7	8	1	2	3	4	5	6	3	4	5	6	7
3	4	5	6	7	8	1	2	7	8	1	2	3	4	5	6	7	8	1	2	7	8	1	2
6	7	8	1	2	3	4	5	6	3	4	5	6	7	8	1	2	3	4	5	6	3	4	5
1	2	3	4	5	6	7	8	1	2	7	8	1	2	3	4	5	6	7	8	1	2	7	8
4	5	6	7	8	1	2	3	4	5	6	3	4	5	6	7	8	1	2	3	4	5	6	3
7	8	1	2	3	4	5	6	7	8	1	2	7	8	1	2	3	4	5	6	7	8	1	2
6	3	4	5	6	7	8	1	2	3	4	5	6	3	4	5	6	7	8	1	2	3	4	5
1	2	7	8	1	2	3	4	5	6	7	8	1	2	7	8	1	2	3	4	5	6	7	8

Fig. 7. Example of perfect sphere packing using Construction 2.1 and perfect t-interleaving using Construction 2.2.

We comment that Construction 2.2 provides the *complete* set of perfect t-interleaving constructions that have the following property: for any two integers, the two sets of vertices respectively labelled by those two integers are cosets of each other in the torus. What is more, in [11], three t-interleaving constructions for two-dimensional arrays were presented, all based on lattice interleavers. Those three constructions can also be applied to tori because of their periodic patterns. Our Construction 2.2 generalizes the results in [11] in two ways: firstly, it covers more constructions based on lattice interleavers, with the results of [11] included as special cases; secondly, when t is even, it also covers constructions that do not use lattice interleavers, which we can make happen by simply letting any A_i and A_j take different values.

III. Achieving an Interleaving Degree within One of the Optimal

In this section, we present a novel t-interleaving construction, with which we can t-interleave any large enough torus with a degree within one of the optimal. The construction presented here will also be used as a building block in Section IV for optimal t-interleaving.

A. Interleaving Construction

Definition 3.1:

- Given a positive integer t, if t is odd, then P is defined to be a string of integers 'a_1, a_2, $\cdots$, $a_{\frac{t-1}{2}}$', where $a_{\frac{t-1}{2}} = t+1$ and $a_i = t$ for $1 \leq i < \frac{t-1}{2}$; if t is even, then P is defined to be a string of integers 'a_1, a_2, $\cdots$, $a_{\frac{t}{2}}$', where $a_{\frac{t}{2}} = t$ and $a_i = t-1$ for $1 \leq i < \frac{t}{2}$. (For example, if $t=3$, then $P=$'4'; if $t=4$, then $P=$'3,4'; if $t=5$, then $P=$'5,6'.)
- Given a positive integer t, if t is odd, then Q is defined to be a string of integers 'b_1, b_2, $\cdots$, $b_{\frac{t+1}{2}}$', where $b_{\frac{t+1}{2}} = t+1$ and $b_i = t$ for $1 \leq i < \frac{t+1}{2}$; if t is even, then Q is defined to be a string of integers 'b_1, b_2, $\cdots$, $b_{\frac{t}{2}+1}$', where $b_{\frac{t}{2}+1} = t$ and $b_i = t-1$ for $1 \leq i < \frac{t}{2}+1$.
- Given a positive integer t, an *offset sequence* is a string of 'P's and 'Q's. (As an example, an offset sequence consisting of 1 'P' and 2 'Q's can be 'PQQ', 'QPQ' or 'QQP'.) The offset sequence is also naturally seen as a string of integers

0	2	4	0	3	5	1	4
1	3	5	1	4	0	2	5
2	4	0	2	5	1	3	0
3	5	1	3	0	2	4	1
4	0	2	4	1	3	5	2
5	1	3	5	2	4	0	3

Fig. 8. An example of t-interleaving with the three features.

which is the union of the integers in its 'P's and 'Q's. (For example, when $t = 3$, if an offset sequence consisting of 1 'P' and 2 'Q's is 'PQQ', then the offset sequence is also seen as '4,3,4,3,4'; when $t = 4$, if an offset sequence consisting of 3 'P's and 2 'Q's is '$PQPPQ$', then the offset sequence is also seen as '3,4,3,3,4,3,4,3,4,3,3,4'.) The number of integers in an offset sequence is called its *length*.

□

In this section, we are particularly interested in one kind of t-interleaving on an $l_1 \times l_2$ torus, which has the following features:

- Feature 1: $l_1 = |S_t| + 1$. (In other words, if t is odd, then $l_1 = \frac{t^2+1}{2} + 1$; if t is even, then $l_1 = \frac{t^2}{2} + 1$.)
- Feature 2: The degree of the t-interleaving equals l_1. And in every column of the torus, each of the l_1 integers is assigned to exactly one vertex.
- Feature 3: If the vertex (a_1, b_1) and the vertex (a_2, b_2) are labelled by the same integer, then for $i = 1, 2, \cdots, l_1 - 1$, the vertex $((a_1 + i) \bmod l_1, b_1)$ and the vertex $((a_2 + i) \bmod l_1, b_2)$ are labelled by the same integer.

Example 3.1: Fig. 8 shows a t-interleaving on an $l_1 \times l_2$ torus which has the above three features. There $t = 3$, $l_1 = |S_t| + 1 = 6$ and $l_2 = 8$.

Now let's fixed an integer 'i', where $0 \leq i \leq 5$, and say the set of vertices labelled by 'i' are '$(x_0, 0)$, $(x_1, 1)$, $\cdots$, $(x_{l_2-1}, l_2 - 1)$'. Then the following string of integers: '$(x_1 - x_0) \bmod l_1, (x_2 - x_1) \bmod l_1, \cdots, (x_7 - x_6) \bmod l_1, (x_0 - x_7) \bmod l_1$', equals '4,4,4,3,4,4,3,4'. Since when $t = 3$, $P =$'4' and $Q =$'3,4', the above string of integers actually equals '$PPPQPQ$', which is an offset sequence of length l_2. We comment that this phenomenon is not a pure coincidence — offset sequences do help us find t-interleavings that have the above three features. In fact, we can prove that in many cases (e.g., when $t = 5$ or 7), for *any* t-interleaving on a torus that has the above three features, after horizontally shifting and/or vertically reversing the interleaving pattern, the resulting interleaving will have the same phenomenon as the example shown here.

□

The following construction outputs a t-interleaving that has the three features.

Construction 3.1:

Input: A positive integer t. An $l_1 \times l_2$ torus, where $l_1 = |S_t| + 1$. An integer m that equals $\lfloor \frac{t}{2} \rfloor$. Two integers p and q that satisfy the following equation set if t is odd:

$$\begin{cases} pm + q(m+1) = l_2 \\ p(2m^2 + m + 1) + q(2m^2 + 3m + 2) \equiv 0 \bmod (2m^2 + 2m + 2) \\ p \text{ and } q \text{ are non-negative integers, } p + q > 0. \end{cases} \tag{1}$$

and satisfy the following equation set if t is even:

$$\begin{cases} pm + q(m+1) = l_2 \\ p(2m^2 - m + 1) + q(2m^2 + m) \equiv 0 \bmod (2m^2 + 1) \\ p \text{ and } q \text{ are non-negative integers, } p + q > 0. \end{cases} \tag{2}$$

Output: A t-interleaving on the $l_1 \times l_2$ torus that satisfies Feature 1, Feature 2 and Feature 3.

Construction: Let $S =$‘$s_0, s_1, \cdots, s_{l_2-1}$’ be an arbitrary offset sequence consisting of p ‘P’s and q ‘Q’s. For $j = 1, 2, \cdots, l_2$ and for $i = 0, 1, \cdots, l_1 - 1$, label the vertex $((\sum_{k=0}^{j-1} s_k + i) \bmod l_1, j \bmod l_2)$ with the integer ‘i’.

□

Example 3.2: Let $t = 3$, $l_1 = 6$, $l_2 = 8$, $m = 1$, $p = 4$, and $q = 2$. We use Construction 3.1 to t-interleave an $l_1 \times l_2$ torus. Say the offset sequence S is chosen to be ‘$PPPQPQ$’. Then Construction 3.1 outputs the t-interleaving shown in Fig. 8. □

We explain Construction 3.1 a little bit. The Equation Set (1) (for odd t) and the Equation Set (2) (for even t) ensure that the offset sequence S, which consists of p ‘P’s and q ‘Q’s, exists. Furthermore, for any integer j ($0 \le j \le l_2 - 1$), if (a, j) and $(b, (j+1) \bmod l_2)$ are two vertices labelled by the same integer, then $b - a \equiv s_j \bmod l_1$ — namely, the offset sequence S indicates the *vertical offsets* of any two vertices in adjacent columns that are labelled by the same integer. It is simple to verify that the t-interleaving output by Construction 3.1 satisfies all the three features — Feature 1, 2 and 3 — listed earlier in this subsection.

The following lemma will be used to prove the correctness of Construction 3.1 and also in future analysis.

Lemma 4: Let $i \in \{0, 1, \cdots, |S_t|\}$ be any of the integers used by Construction 3.1 to interleave the $l_1 \times l_2$ torus. Let $\{(b_0, 0), (b_1, 1), \cdots, (b_{l_2-1}, l_2 - 1)\}$ be the set of vertices in the torus that are labelled by i. Let m and S have the same meaning as in Construction 3.1 (namely, $m = \lfloor \frac{t}{2} \rfloor$, and $S =$‘$s_0, s_1, \cdots, s_{l_2-1}$’ is the offset sequence consisting of p ‘P’s and q ‘Q’s utilized by Construction 3.1). For any two integers j_1 and j_2 ($0 \le j_1 \ne j_2 \le l_2 - 1$), we define $L_{j_1 \to j_2}$ as $L_{j_1 \to j_2} = [(j_2 - j_1) \bmod l_2] + \min\{(b_{j_2} - b_{j_1}) \bmod l_1, (b_{j_1} - b_{j_2}) \bmod l_1\}$. Then we have the following conclusions:

- Case 1: t is odd, $j_2 - j_1 \equiv m \bmod l_2$, and $s_{j_1}, s_{(j_1+1) \bmod l_2}, s_{(j_1+2) \bmod l_2}, \cdots, s_{(j_2-1) \bmod l_2}$ do not all equal t. In this case, $b_{j_2} - b_{j_1} \equiv -(m+1) \bmod l_1$ and $L_{j_1 \to j_2} = t$.
- Case 2: t is odd, $j_2 - j_1 \equiv m + 1 \bmod l_2$, and exactly one of $s_{j_1}, s_{(j_1+1) \bmod l_2}, s_{(j_1+2) \bmod l_2}, \cdots, s_{(j_2-1) \bmod l_2}$ equals $t + 1$. In this case, $b_{j_2} - b_{j_1} \equiv m \bmod l_1$ and $L_{j_1 \to j_2} = t$.
- Case 3: t is even, $j_2 - j_1 \equiv 1 \bmod l_2$, and $s_{j_1} = t - 1$. In this case, $b_{j_2} - b_{j_1} \equiv t - 1 \bmod l_1$ and $L_{j_1 \to j_2} = t$.
- Case 4: t is even, $j_2 - j_1 \equiv m \bmod l_2$, and $s_{j_1}, s_{(j_1+1) \bmod l_2}, s_{(j_1+2) \bmod l_2}, \cdots, s_{(j_2-1) \bmod l_2}$ do not all equal $t - 1$. In this case, $b_{j_2} - b_{j_1} \equiv -m \bmod l_1$ and $L_{j_1 \to j_2} = t$.
- Case 5: t is even, $j_2 - j_1 \equiv m + 1 \bmod l_2$, and exactly one of $s_{j_1}, s_{(j_1+1) \bmod l_2}, s_{(j_1+2) \bmod l_2}, \cdots, s_{(j_2-1) \bmod l_2}$ equals t. In this case, $b_{j_2} - b_{j_1} \equiv m - 1 \bmod l_1$ and $L_{j_1 \to j_2} = t$.
- If none of the above five cases is true, and $j_2 - j_1 \ne t \bmod l_2$, then $L_{j_1 \to j_2} > t$. If none of the above five cases is true, and $j_2 - j_1 \equiv t \bmod l_2$, then $L_{j_1 \to j_2} \ge t$.

Proof: Let $\Delta = t + 1$ if t is odd, and let $\Delta = t$ if t is even. The offset sequence S consists of ‘P’s and ‘Q’s, so it has the following property: for any $k \in \{0, 1, \cdots, l_2 - 1\}$ such that $s_k = \Delta$, the following $m - 1$ integers — $s_{(k+1) \bmod l_2}$, $s_{(k+2) \bmod l_2}$, $\cdots$, $s_{(k+m-1) \bmod l_2}$ — all equal $\Delta - 1$, and either $s_{(k+m) \bmod l_2}$ or $s_{(k+m+1) \bmod l_2}$ equals Δ. Also note that $b_{j_2} - b_{j_1} \equiv s_{j_1} + s_{(j_1+1) \bmod l_2} + s_{(j_1+2) \bmod l_2} + \cdots + s_{(j_2-1) \bmod l_2} \bmod l_1$. Based on those two observations, this lemma can be proved with straightforward computation.

□

Theorem 5: Construction 3.1 is correct.

Proof: Let (b_{j_1}, j_1) and (b_{j_2}, j_2) be any two vertices labelled by the same integer in the $l_1 \times l_2$ torus that was interleaved by Construction 3.1. The Lee distance between them is $d((b_{j_1}, j_1), (b_{j_2}, j_2)) = \min\{(j_2 - j_1) \bmod l_2, (j_1 - j_2) \bmod l_2\} + \min\{(b_{j_2} - b_{j_1}) \bmod l_1, (b_{j_1} - b_{j_2}) \bmod l_1\} = \min\{L_{j_1 \to j_2}, L_{j_2 \to j_1}\}$. From Lemma 4, it is clearly that neither $L_{j_1 \to j_2}$ nor $L_{j_2 \to j_1}$ is less than t. Therefore $d((b_{j_1}, j_1), (b_{j_2}, j_2)) \ge t$. So Construction 3.1 t-interleaved the torus. And as mentioned before, this t-interleaving satisfies Feature 1, Feature 2 and Feature 3.

□

B. Existence of Offset Sequences

The feasibility of Construction 3.1 depends only on one thing — whether the two input parameters 'p' and 'q' exist or not. The following theorem shows that when the width of the torus, l_2, exceeds a threshold, 'p' and 'q' are guaranteed to exist.

Theorem 6: Let t be an odd (respectively, even) positive integer. When $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t| + 1)$, there exists at least one solution (p, q) to the equation set (1) (respectively, equation set (2)), which is shown in the 'Input' part of Construction 3.1.

Proof: Firstly, let's assume t is odd. The equation set (1) is as follows:

$$\begin{cases} pm + q(m+1) = l_2 \\ p(2m^2+m+1) + q(2m^2+3m+2) \equiv 0 \bmod (2m^2+2m+2) \\ p \text{ and } q \text{ are non-negative integers, } p+q > 0. \end{cases}$$

where $m = \lfloor \frac{t}{2} \rfloor$. We introduce a new variable z, and transform the above equation set equivalently to be:

$$\begin{cases} \begin{pmatrix} m & m+1 \\ 2m^2+m+1 & 2m^2+3m+2 \end{pmatrix} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} l_2 \\ z(2m^2+2m+2) \end{pmatrix} \\ p \text{ and } q \text{ are non-negative integers; } z \text{ is a positive integer.} \end{cases}$$

which is the same as:

$$\begin{cases} \begin{pmatrix} p \\ q \end{pmatrix} = \begin{pmatrix} m & m+1 \\ 2m^2+m+1 & 2m^2+3m+2 \end{pmatrix}^{-1} \begin{pmatrix} l_2 \\ z(2m^2+2m+2) \end{pmatrix} \\ p \text{ and } q \text{ are non-negative integers; } z \text{ is a positive integer.} \end{cases}$$

which equals:

$$\begin{cases} p = 2(m+1)(m^2+m+1)z - (2m^2+3m+2)l_2 \\ q = (2m^2+m+1)l_2 - 2m(m^2+m+1)z \\ p \text{ and } q \text{ are non-negative integers; } z \text{ is a positive integer.} \end{cases}$$

There exists a solution for the variables p, q and z in the above equation set if and only if the following conditions can be satisfied:

$$\begin{cases} 2(m+1)(m^2+m+1)z - (2m^2+3m+2)l_2 \geq 0 \\ (2m^2+m+1)l_2 - 2m(m^2+m+1)z \geq 0 \\ z \text{ is a positive integer.} \end{cases}$$

which is equivalent to:

$$\begin{cases} \frac{(2m^2+3m+2)l_2}{2(m+1)(m^2+m+1)} \leq z \leq \frac{(2m^2+m+1)l_2}{2m(m^2+m+1)} \\ z \text{ is a positive integer.} \end{cases}$$

To enable a value for z to exist that satisfies the above conditions, it is sufficient to make $\frac{(2m^2+m+1)l_2}{2m(m^2+m+1)} - \frac{(2m^2+3m+2)l_2}{2(m+1)(m^2+m+1)} \geq 1$ — that is, to make $l_2 \geq 2m(m+1)(m^2+m+1) = \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$. Therefore when $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, there exists at least one solution (p, q) to the equation set (1).

When t is even, the conclusion can be proved in a very similar way. We skip its details.
□

Corollary 1: When $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, Construction 3.1 can be used to output a t-interleaving on an $(|S_t|+1) \times l_2$ torus.

Proof: When $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, all the parameters in the 'Input' part of Construction 3.1 exist, including p and q.
□

(a)

A

0	1	2	3
3	2	1	0

B

0	0	0	0
1	1	1	1
2	2	2	2

C

1	2
3	4

(b)

D

0	1	2	3
3	2	1	0
0	0	0	0
1	1	1	1
2	2	2	2

E

1	2	0	1	2	3	1	2
3	4	3	2	1	0	3	4

Fig. 9. Examples of *tiling tori*

C. *Interleaving with Degree within One of the Optimal*

In this subsection, we will show how to interleave a large enough torus with the degree within one of the optimal.

We define the simple term of *tiling tori* here. By tiling several interleaved tori vertically or horizontally, we get a larger torus, whose interleaving is the straightforward combination of the interleaving on the smaller tori. It is best explained with an example.

Example 3.3: Three interleaved tori— A, B and C — are shown in Fig.9. The torus D is a 5×4 torus, got by *tiling A and B vertically* in the form of $\begin{bmatrix} A \\ B \end{bmatrix}$. The torus E is a 2×8 torus, got by *tiling one copy of A and two copies of C horizontally* in the form of $\begin{bmatrix} C & A & C \end{bmatrix}$.
$\square$

The following construction t-interleaves a large enough torus with at most $|S_t| + 2$ distinct integers.

Construction 3.2: t-interleave an $l_1 \times l_2$ torus G, where $l_1 \geq |S_t|(|S_t| + 1)$ and $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t| + 1)$, using at most $|S_t| + 2$ distinct integers.

1. Let G_1 be an $(|S_t| + 1) \times l_2$ torus that is t-interleaved by Construction 3.1, using the integers '0','1', $\cdots$, '$|S_t|$'. Let $\{(c_0, 0), (c_1, 1), \cdots, (c_{l_2-1}, l_2 - 1)\}$ be the set of vertices in G_1 labelled by the integer '0'.
2. Let G_2 be an $(|S_t|+2) \times l_2$ torus. Label the vertices $\{(c_0, 0), (c_1, 1), \cdots, (c_{l_2-1}, l_2 - 1)\}$ *in* G_2 with the integer '$|S_t|+1$'.
3. For $j = 0, 1, \cdots, l_2 - 1$ and for $i = 1, 2, \cdots, |S_t| + 1$, label the vertex $((c_j + i) \bmod (|S_t| + 2), j)$ *in* G_2 with the integer '$i - 1$'.
4. Let x and y be two non-negative integers such that $l_1 = x(|S_t| + 1) + y(|S_t| + 2)$. Tile x copies of G_1 and y copies of G_2 vertically to get an $l_1 \times l_2$ torus G. (Then G has been t-interleaved using at most $|S_t| + 2$ distinct integers.)

$\square$

Example 3.4: We use Construction 3.2 to t-interleave a 7×6 torus G, where $t = 2$. The first step is to use Construction 3.1 to t-interleave a 3×6 torus G_1. Say the offset sequence selected in Construction 3.1 is $S =$ '$QQQ' =$ '$1, 2, 1, 2, 1, 2'$, then G_1 is as shown in Fig. 10. Then the 4×6 torus G_2 is as shown in the figure. By tiling one copy of G_1 and one copy of G_2 vertically, we get the t-interleaved torus G. $|S_t| + 2 = 4$ distinct integers are used to interleave G.
$\square$

Theorem 7: Construction 3.2 is correct.

Proof: It is a known fact that for any two relatively prime positive integers A and B, any integer C no less than $(A-1)(B-1)$ can be expressed as $C = xA + yB$ where x and y are non-negative integers. Therefore in Construction 3.2, since $l_1 \geq |S_t|(|S_t| + 1)$, l_1 indeed can be expressed as $l_1 = x(|S_t| + 1) + y(|S_t| + 2)$, as shown in the last step of Construction 3.2. So the construction can be executed from beginning to end successfully. Now we prove that the construction does t-interleave G — that is, for any two vertices (a_1, b_1) and (a_2, b_2) labelled by the same integer i in G, the Lee distance between them is at least t. We consider three cases.

G_1

0	2	0	2	0	2
1	0	1	0	1	0
2	1	2	1	2	1

G_2

3	2	3	2	3	2
0	3	0	3	0	3
1	0	1	0	1	0
2	1	2	1	2	1

G

0	2	0	2	0	2
1	0	1	0	1	0
2	1	2	1	2	1
3	2	3	2	3	2
0	3	0	3	0	3
1	0	1	0	1	0
2	1	2	1	2	1

Fig. 10. Examples of Construction 3.2.

Case 1: $b_1 = b_2$, which means that (a_1, b_1) and (a_2, b_2) are in the same column of G. We see every column of G as a ring of length l_1 (because it is toroidal). Then, observe the integers labelling a column of G, and we can see that on the column, the integers following an integer '$|S_t|+1$' and before the next integer '$|S_t|+1$' must be '$0, 1, \cdots, |S_t|, 0, 1, \cdots, |S_t|, \cdots\cdots, 0, 1, \cdots, |S_t|$', where the pattern $0, 1, \cdots, |S_t|$ appears at least once. Therefore since (a_1, b_1) and (a_2, b_2) are labelled by the same integer, the Lee distance between them must be at least $|S_t| + 1 > t$.

Case 2: $b_1 \neq b_2$, and $i \neq |S_t| + 1$. In this case, let's first observe two conclusions:

- The interleaving on G_2 is t-interleaving. (See Construction 3.2 for the definition of G_2.) This can be proved as follows: any two vertices labelled by the same integer in G_2 can be expressed as $((c_{j_1} + i_0) \bmod (|S_t| + 2), j_1)$ and $((c_{j_2} + i_0) \bmod (|S_t| + 2), j_2)$ (see the Step 2 and Step 3 of Construction 3.2); then, $d_{G_2}(((c_{j_1} + i_0) \bmod (|S_t| + 2), j_1), ((c_{j_2} + i_0) \bmod (|S_t| + 2), j_2)) = d_{G_2}((c_{j_1}, j_1), (c_{j_2}, j_2)) \geq d_{G_1}((c_{j_1}, j_1), (c_{j_2}, j_2)) \geq t$.
- Let (α, j) and (β, j) be two vertices respectively in G_1 and G_2 both of which are labelled by the same integer. Then it is simple to see that $\beta = \alpha$ or $\beta = \alpha + 1$. Since G_1 has $|S_t| + 1$ rows and G_2 has $|S_t| + 2$ rows, we have $d_{G_2}((\beta, j), (0, j)) \geq d_{G_1}((\alpha, j), (0, j))$ and $d_{G_2}((\beta, j), (|S_t| + 1, j)) \geq d_{G_1}((\alpha, j), (|S_t|, j))$. That is, if u and v are two vertices respectively in G_1 and G_2 both of which are in the j-th column and labelled by the same integer, the vertical distance from v to the two 'borders' of G_2 is no less than the vertical distance from u to the two 'borders' of G_1.

According to Construction 3.2, G is got by vertically tiling x copies of G_1 and y copies of G_2. Let's call each of those $x + y$ tori a *component torus* of G. Now, if (a_1, b_1) and (a_2, b_2) are in the same component torus of G, we know the Lee distance between them in G is no less than the Lee distance between them in that component torus, which is at least t because that component torus is t-interleaved. If (a_1, b_1) and (a_2, b_2) are not in the same component torus of G, we do the following. We firstly construct a torus G' which is got by vertically tiling $x + y$ copies of G_1. It is simple to see that G' is t-interleaved. We call each of the $x + y$ copies of G_1 in G' a *component torus* of G'. Let's say (a_1, b_1) and (a_2, b_2) are respectively in the k_1-th and k_2-th component torus of G. Let (c_1, b_1) and (c_2, b_2) be the two vertices labelled by the integer i that are respectively in the k_1-th and k_2-th component torus of G'. Observe the shortest path between (a_1, b_1) and (a_2, b_2) in G, and we see that it can be split into such three intervals: from (a_1, b_1) to a border of the k_1-th component torus, from the border of the k_1-th component torus to the border of the k_2-th component torus, and from the border of the k_2-th component torus to (a_2, b_2). There is a corresponding (not necessarily shortest) path connecting (c_1, b_1) and (c_2, b_2) in G', which can be split into such three intervals similarly. And each of the three intervals of the first path is at least as long as the corresponding interval of the second path. G' is t-interleaved, so the second path's length is at least t. So the Lee distance between (a_1, b_1) and (a_2, b_2) in G is at least t.

Case 3: $b_1 \neq b_2$, and $i = |S_t| + 1$. In this case, it is simple to see that the two vertices in G, $(a_1 + 1 \bmod l_1, b_1)$ and $(a_2 + 1 \bmod l_1, b_2)$, are both labelled by the integer 0. Based on the conclusion of Case 2, $d_G((a_1 + 1 \bmod l_1, b_1), (a_2 + 1 \bmod l_1, b_2)) \geq t$. So $d_G((a_1, b_1), (a_2, b_2)) = d_G((a_1 + 1 \bmod l_1, b_1), (a_2 + 1 \bmod l_1, b_2)) \geq t$.

So Construction 3.2 correctly t-interleaved G.

□

As a result of Construction 3.2, we get the following theorem.

Theorem 8: When $l_1 \geq |S_t|(|S_t|+1)$ and $l_2 \geq \lfloor \frac{t}{2} \rfloor (\lfloor \frac{t}{2} \rfloor + 1)(|S_t|+1)$, an $l_1 \times l_2$ (or equivalently, $l_2 \times l_1$) torus' t-interleaving number is at most $|S_t| + 2$.

By combining Construction 2.2 (the construction for perfect t-interleaving) and Construction 3.2, we can t-interleave any sufficiently large torus with a degree within one of the optimal.

IV. Optimal Interleaving on Large Tori

In the previous section, it is shown that when l_2 is large enough, an $(|S_t| + 1) \times l_2$ torus can be t-interleaved using $|S_t| + 1$ integers. In this section, we will construct an $[k(|S_t| + 1) - 1] \times l_2$ torus which is also t-interleaved using $|S_t| + 1$ integers, by using an operation we call '*removing a zigzag row*'. ('k' is some integer.) Those two tori have a special property: when they (or multiple copies of them) are tiled vertically to get a larger torus, the larger torus is also t-interleaved with degree $|S_t| + 1$. $|S_t| + 1$ and $k(|S_t| + 1) - 1$ are relatively prime, so a large enough l_1 must be a linear combination of those two numbers with non-negative integral coefficients — therefore an $l_1 \times l_2$ torus can be t-interleaved using $|S_t| + 1$ integers in this way. We present constructions to optimally t-interleave such tori; and as a parallel result, the existence of Region I (see Section I: Introduction) is proved.

All the results of this section can be split into two parts: one for the case 't is odd', and the other for the case 't is even'. Those two cases can be analyzed with very similar methods; however their analysis and results differ in details. For succinctness, in this section, we only analyze in detail the case 't is odd', which should suffice for illustrating all the ideas. So in the first three subsections here — Subsection A, B, and C, we always assume that *t is odd*. In Subsection D, we present just the final result for the case 't is even'. We list the major intermediate results for the case 't is even' in Appendix II.

A. Removing a Zigzag Row in a Torus

Definition 4.1: A *zigzag row* in an $l_1 \times l_2$ torus is a set of l_2 vertices of the torus: $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, where $0 \leq a_i \leq l_1 - 1$ for $i = 0, 1, \cdots, l_2 - 1$. (For example, $\{(2, 0), (3, 1), (0, 2), (0, 3), (3, 4)\}$ is a zigzag row in a 4×5 torus.) $\square$

Definition 4.2: Let T be an $l_1 \times l_2$ torus. Let $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$ be a zigzag row in T. Let there be an interleaving on T, which labels T's vertex (b, c) with the integer $I(b, c)$, for $b = 0, 1, \cdots, l_1 - 1$ and $c = 0, 1, \cdots, l_2 - 1$. Then a torus G is said to be 'got by removing the zigzag row $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$ in T' if and only if these two conditions are satisfied:

- G is an $(l_1 - 1) \times l_2$ torus.
- For $i = 0, 1, \cdots, l_1 - 2$ and $j = 0, 1, \cdots, l_2 - 1$, the vertex (i, j) in G is labelled by the integer $I(i, j)$ if $i < a_j$, and by the integer $I(i + 1, j)$ if $i \geq a_j$. $\square$

Example 4.1: In Fig. 11, a 6×5 torus T is shown. A zigzag row $\{(3, 0), (2, 1), (1, 2), (3, 3), (1, 4)\}$ in T is circled in the figure. Fig. 11 shows a torus G got by removing the zigzag row $\{(3, 0), (2, 1), (1, 2), (3, 3), (1, 4)\}$ in T.

It can be readily observed that G can be seen as being derived from T in the following way: firstly, delete the zigzag row in T that is circled in Fig. 11; then in each column of T, move the vertices below the circled vertex upward. $\square$

In order to get our final results, we present three rules to follow for devising a zigzag row. Let B be an $l_0 \times l_2$ torus which is t-interleaved by Construction 3.1. (That means $l_0 = |S_t| + 1$.) Let $S =$'$s_0, s_1, \cdots, s_{l_2-1}$' be the offset sequence utilized by Construction 3.1 when it was t-interleaving B. Let H be an $l_1 \times l_2$ torus got by tiling several copies of B vertically. Let $m = \lfloor \frac{t}{2} \rfloor$. Then the three rules for devising a zigzag row <u>in H</u> — $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$ — are:

T

1	3	5	2	4
2	4	6	3	5
3	5	1	4	6
4	6	2	5	1
5	1	3	6	2
6	2	4	1	3

G

1	3	5	2	4
2	4	1	3	6
3	6	2	4	1
5	1	3	6	2
6	2	4	1	3

Fig. 11. Removing a zigzag row $\{(3,0),(2,1),(1,2),(3,3),(1,4)\}$ in T.

- Rule 1: For any j such that $0 \le j \le l_2 - 1$, if the integers $s_j, s_{(j+1) \bmod l_2}, \cdots, s_{(j+m-1) \bmod l_2}$ do not all equal t, then $a_j \ge a_{(j+m) \bmod l_2} + m$.
- Rule 2: For any j such that $0 \le j \le l_2 - 1$, if exactly one of the integers $s_j, s_{(j+1) \bmod l_2}, \cdots, s_{(j+m) \bmod l_2}$ equals $t+1$, then $a_j \le a_{(j+m+1) \bmod l_2} - (m-1)$.
- Rule 3: For any j such that $0 \le j \le l_2 - 1$, $m \le a_j \le l_1 - m - 1$.

Lemma 5: Let B be a torus t-interleaved by Construction 3.1. Let H be a torus got by tiling copies of B vertically, and let T be a torus got by removing a zigzag row in H, where the zigzag row in H follows the three rules — Rule 1, Rule 2 and Rule 3. Let G be a torus got by tiling copies of B and T vertically. Then, both T and G are t-interleaved.

Proof: When $t = 1$, the proof is trivial. So we assume $t \ge 3$ in the rest of the proof. It is simple to see that H is t-interleaved, because H is got by tiling B, a t-interleaved torus. We assume B is an $l_0 \times l_2$ torus (where $l_0 = |S_t| + 1$), H is an $l_1 \times l_2$ torus (where l_1 is a multiple of l_0), T is an $l_T \times l_2$ torus (where $l_T = l_1 - 1$), and G is an $l_G \times l_2$ torus. Let $m = \lfloor \frac{t}{2} \rfloor$. Let $S =$'$s_0, s_1, \cdots, s_{l_2-1}$' be the offset sequence utilized by Construction 3.1 when it was t-interleaving B.

(1) In this part, we will prove that T is t-interleaved. Let (x_1, y_1) and (x_2, y_2) be two vertices in T both labelled by some integer 'r'. We need to prove that $d_T((x_1,y_1),(x_2,y_2)) \ge t$.

Let $\{(a_0,0),(a_1,1),\cdots,(a_{l_2-1}, l_2-1)\}$ denote the zigzag row removed in H to get T. If $a_{y_1} \le x_1$, then let $z_1 = x_1 + 1$; otherwise let $z_1 = x_1$. Similarly, if $a_{y_2} \le x_2$, then let $z_2 = x_2 + 1$; otherwise let $z_2 = x_2$. Clearly, the two vertices in H, (z_1, y_1) and (z_2, y_2), are also labelled by 'r'.

We only need to consider the following three cases:

Case 1: $y_1 = y_2$. In this case, $d_H((z_1,y_1),(z_2,y_2))$ is a multiple of $|S_t|+1$ (the number of rows in B); and $d_T((x_1,y_1),(x_2, y_2)) \ge d_H((z_1,y_1),(z_2,y_2)) - 1 \ge |S_t| = \frac{t^2+1}{2} > t$.

Case 2: $y_1 \ne y_2$ and $d_T((x_1,y_1),(x_2,y_2)) \le d_H((z_1,y_1),(z_2,y_2)) - 2$. Without loss of generality (WLOG), we assume $x_1 \ge x_2$. Then, based on the definition of the 'removing a zigzag row', it is simple to verify that the following must be true: $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) - 2$, $a_{y_2} < z_2 < z_1 < a_{y_1}$, $(z_2 - z_1 \bmod l_1) \le (z_1 - z_2 \bmod l_1)$. By Rule 3, any vertex in the removed zigzag row is neither in the first m rows nor in the last m rows of H, so $(z_2 - z_1 \bmod l_1) \ge 2m+3$. So $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) - 2 > (z_2 - z_1 \bmod l_1) - 2 \ge 2m+1 = t$.

Case 3: $y_1 \ne y_2$ and $d_T((x_1,y_1),(x_2,y_2)) \ge d_H((z_1,y_1),(z_2,y_2)) - 1$. We know that $d_H((z_1,y_1),(z_2,y_2)) \ge t$. So to show that $d_T((x_1,y_1),(x_2,y_2)) \ge t$, we just need to prove that if $d_H((z_1,y_1),(z_2,y_2)) = t$, then $d_T((x_1,y_1),(x_2,y_2)) \ge d_H((z_1,y_1),(z_2,y_2))$. By Lemma 4, there are only two non-trivial sub-cases to consider WLOG:

Sub-case 3.1: $y_2 - y_1 \equiv m \bmod l_2$, $z_2 - z_1 \equiv -(m+1) \bmod l_1$, $d_H((z_1,y_1),(z_2,y_2)) = (y_2 - y_1 \bmod l_2) + (z_1 - z_2 \bmod l_1) = t$, and s_{y_1}, $s_{(y_1+1) \bmod l_2}$, $s_{(y_1+2) \bmod l_2}$, $\cdots$, $s_{(y_1+m-1) \bmod l_2}$ do not all equal t. If $z_1 > z_2$ (which means $z_1 = z_2 + (m+1)$), then from Rule 1, it is simple to see that $x_1 - x_2 = z_1 - z_2$ — so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$. If $z_1 < z_2$ (which means that (z_1,y_1) and (z_2,y_2) are respectively in the first and last $m+1$ rows of H), since the first and last m rows of H and T must be the same, we get that $(x_1 - x_2 \bmod l_T) = (z_1 - z_2 \bmod l_1) = m+1$ – so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$.

Sub-case 3.2: $y_2 - y_1 \equiv m+1 \bmod l_2$, $z_2 - z_1 \equiv m \bmod l_1$, $d_H((z_1,y_1),(z_2,y_2)) = (y_2 - y_1 \bmod l_2) + (z_2 - z_1 \bmod l_1) = t$, and exactly one of s_{y_1}, $s_{(y_1+1) \bmod l_2}$, $s_{(y_1+2) \bmod l_2}$, $\cdots$, $s_{(y_1+m) \bmod l_2}$ equals $t+1$. If $z_1 < z_2$ (which means $z_1 = z_2 - m$), then from Rule 2, it is simple to see that $x_2 - x_1 = z_2 - z_1$ — so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$. If $z_1 > z_2$ (which means that (z_1,y_1) and (z_2,y_2) are respectively in the last and first m rows of H), since the first and last m rows of H and T must be the same, we get that $(x_2 - x_1 \bmod l_T) = (z_2 - z_1 \bmod l_1) = m$ — so $d_T((x_1,y_1),(x_2,y_2)) = d_H((z_1,y_1),(z_2,y_2)) = t$.

So T is t-interleaved.

(2) In this part, we will prove that G is t-interleaved. First let's have an observation: when a t-interleaved torus K is tiled with other tori vertically to get a larger torus $\hat{G}$, for any two vertices μ and ν in K (which are now also in $\hat{G}$) labelled by the same integer, the Lee distance between them $\underline{\text{in } \hat{G}}$, $d_{\hat{G}}(\mu,\nu)$, is clearly no less than t. Let's also notice that the torus got by tiling one copy of B and one copy of T vertically is t-interleaved, which can be proved with exactly the same proof as in part (1).

G is got by tiling multiple copies of B and T. Let's call each copy of B or T in G a *component torus*. Let (x_1,y_1) and (x_2,y_2) be two vertices in G labelled by the same integer. Assume $d_G((x_1,y_1),(x_2,y_2)) \le t$. Then since both B and T have more than t rows, (x_1,y_1) and (x_2,y_2) must be either in the same component torus or in two adjacent component tori. Now if (x_1,y_1) and (x_2,y_2) are in the same component torus, let K denote that component torus; if (x_1,y_1) and (x_2,y_2) are in two adjacent component tori, let K be the torus got by vertically tiling those two component tori; let $\hat{G}$ be the same as G. By using the observation in the previous paragraph, we can readily prove that $d_G((x_1,y_1),(x_2,y_2)) \ge t$. So G is t-interleaved.

□

B. *Constructing the Zigzag Row*

We presented three rules on devising a zigzag row in the previous subsection. But specifically, how to construct a zigzag row that follow all those rules? In this subsection, we present such constructions.

Before the formal presentation, let us go over a few concepts. An offset sequence is a string of 'P's and 'Q's, where P and Q are strings of integers depending on t. For example, when $t = 5$, $P =$'$5,6$' and $Q =$'$5,5,6$'. Then an offset sequence 'PPQ' can also be written as '$5,6,5,6,5,5,6$'. Let's also express the offset sequence 'PPQ' as '$s_0, s_1, s_2, s_3, s_4, s_5, s_6$', where $s_0 = 5$, $s_1 = 6$, $\cdots$, $s_6 = 6$. Then for $i = 0,1,\cdots,6$, s_i is called the '$(i+1)$-th element' of the offset sequence. s_2 is also called the 'first element of a P', because it is the first element of the second P in the offset sequence. For the same reason, s_0 is the first element of a P (the first P in the offset sequence), s_1 is the second (or last) element of a P (the first P in the offset sequence), s_4 is the first element of a Q (the first/last/only Q in the offset sequence), and so on.

Now we begin the formal presentation of the constructions. Let B be an $l_0 \times l_2$ torus that is t-interleaved by Construction 3.1. (Therefore $l_0 = |S_t| + 1$.) Let H be an $l_1 \times l_2$ torus got by tiling z copies of B vertically. (Therefore $l_1 = z l_0 = z(|S_t|+1)$.) Let $S =$'$s_0, s_1, \cdots, s_{l_2-1}$' be the offset sequence utilized by Construction 3.1 when it was t-interleaving B. We say that the offset sequence S consists of p 'P's and q 'Q's, where we require $p > 0$ and $q > 0$. We require that in the offset sequence, the 'P's and 'Q's are interleaved very evenly — to be specific, in the offset sequence, between any two nearby 'P's (including between the last 'P' and the first 'P', because we see the offset sequence as being toroidal, so the last 'P' and the first 'P' are also nearby 'P's), there are either $\lceil \frac{q}{p} \rceil$ or $\lfloor \frac{q}{p} \rfloor$ consecutive 'Q's; and between any two nearby 'Q's (including between the last 'Q' and the first 'Q'), there are either $\lceil \frac{p}{q} \rceil$ or $\lfloor \frac{p}{q} \rfloor$ consecutive 'P's. Also, we require the offset sequence to start with a 'P' and to end with a 'Q'. (For example, an offset sequence consisting of 3 'P's and 5 'Q's that satisfies the above requirements is '$PQQPQQPQ$'.) Let $m = \frac{t-1}{2}$. Let $L = m + m\lceil \frac{p}{q} \rceil$ if $p \ge q$, and let $L = m + (m-1)\lceil \frac{q}{p} \rceil$ if $p < q$. We require that $l_1 \ge (\lceil \frac{p}{q} \rceil + 1)m^2 + 2m + 1$ if $p \ge q$, and require that $l_1 \ge (\lceil \frac{q}{p} \rceil + 1)m^2 + m + (2 - \lceil \frac{q}{p} \rceil)$ if $p < q$. Below we present two constructions for constructing a zigzag row $\underline{\text{in } H}$, applicable respectively when $p \ge q$ and when $p < q$. Note that the constructed zigzag row is denoted by $\{(a_0,0),(a_1,1),\cdots,(a_{l_2-1},l_2-1)\}$. Also note that both constructions require $t > 3$. (The analysis for the case '$t = 3$', as a *somewhat* special case, will be presented in Appendix I.)

Construction 4.1: Constructing a zigzag row in H, when t is odd, $t > 3$, and $p \geq q > 0$

1. Let s_{x_1}, s_{x_2}, $\cdots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$, and each s_{x_i} $(1 \leq i \leq p+q)$ is the first element of a 'P' or 'Q' in the offset sequence S.

Let $a_{x_1} = L$. For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a 'Q', let $a_{x_i} = L$.

For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a 'P', then let $a_{x_i} = a_{x_{i-1}} - m$.

2. For $i = 2$ to m and for $j = 1$ to $p+q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L$.

3. Let s_{y_1}, s_{y_2}, $\cdots$, s_{y_q} be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$, and each s_{y_i} $(1 \leq i \leq q)$ is the last element of a 'Q' in the offset sequence S.

For $i = 1$ to q, let $a_{y_i} = mL + m$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

□

The zigzag row constructed by Construction 4.1 has a quite regular structure. We show it with an example.

Example 4.2: We use this example to illustrate Construction 4.1. In this example, $t = 5$, and B is an 14×18 torus as shown in Fig. 12(a). B is t-interleaved by Construction 3.1 by using the offset sequence $S =$'$PPPQPPPQ$'='$5, 6, 5, 6, 5, 6, 5, 5, 6, 5, 6, 5, 6, 5, 6, 5, 5, 6$'. The torus H is shown in Fig. 12(b). H is an 28×18 torus got by tiling 2 copies of B vertically. The rest of the parameters used by Construction 4.1 are $p = 6$, $q = 2$, $m = 2$ and $L = 8$. It is not difficult to verify that the zigzag row in H constructed by Construction 4.1 is $\{(8, 0), (16, 1), (6, 2), (14, 3), (4, 4), (12, 5), (2, 6), (10, 7), (18, 8), (8, 9), (16, 10), (6, 11), (14, 12), (4, 13), (12, 14), (2, 15), (10, 16), (18, 17)\}$. In Fig.12(b), the vertices in the zigzag row are shown in solid-line circles, solid-line hexagons, or dashed-line circles.

Now we briefly analyze the structure of the zigzag row in H. Let us write the offset sequence S as $S =$'$s_0, s_1, \cdots, s_{17}$'. Then for $i = 0, 1, \cdots, 17$, we can see that s_i actually shows the 'offset' between the i-th column and the $(i+1)$-th column of H — in other words, if we shift the integers in the i-th column of H down (toroidally) by s_i units, we get the $(i+1)$-th column of H. So we can think of s_i as 'spanning from the i-th column to the $(i+1)$-th column of H'. And let's say a P or Q in the offset sequence spans the columns that all its elements span. Then, since the offset sequence here is '$PPPQPPPQ$', the ranges each of them spans is as indicated in Fig. 12(b).

Let us observe the vertices in the zigzag row that are in solid-line circles. If we indicate them by (a_{x_1}, x_1), (a_{x_2}, x_2), $\cdots$, $(a_{x_{p+q}}, x_{p+q})$, where $x_1 < x_2 \cdots < x_{p+q}$, then we can see that $s_{x_1}, s_{x_2}, \cdots, s_{x_{p+q}}$ are the 'first elements' of the 'P's and 'Q's in the offset sequence (namely, each of them is the first element of a 'P' or a 'Q' in the offset sequence). And we can see that the vertices in solid-line circles have a regular structure — basically, it climes up by $m = 2$ units from one vertex to the next, and drops to a base-position if it is between the spanned ranges of a Q and a P. Now let us observe the vertices in solid-line hexagons. We can see that they correspond to those 'second elements of the 'P's and 'Q's in the offset sequence', and they also have a regular structure. To be specific, the positions of the vertices in solid-line hexagons can be got by shifting the positions of the vertices in solid-line circles horizontally by 1 unit and then down by $L = 8$ units. In general, those vertices in a zigzag row that correspond to the $(i+1)$-th elements of 'P's and 'Q's can be got by shifting the positions of the vertices that correspond to the i-th elements of 'P's and 'Q's horizontally by 1 unit and down by L unit (here $0 \leq i < m$). As for the vertices in dashed-line circles, they correspond to the 'last elements of the 'Q's in the offset sequence', and they are all in the same row. The above observations can be extended in an obvious way to the general outputs of Construction 4.1.

□

Now we present the second construction.

Construction 4.2: Constructing a zigzag row in H, when t is odd, $t > 3$, and $0 < p < q$

1. Let s_{x_1}, s_{x_2}, $\cdots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$, and each s_{x_i} $(1 \leq i \leq p+q)$ is the first element of a 'P' or 'Q' in the offset sequence S.

Let $a_{x_1} = L$.

For $i = 2$ to $p+q$, if s_{x_i} is the first element of a 'P', let $a_{x_i} = L$; if $s_{x_{i-1}}$ is the first element of a 'P', let $a_{x_i} = L - \lceil \frac{q}{p} \rceil (m-1)$; otherwise, let $a_{x_i} = a_{x_{i-1}} + (m-1)$.

(a) B

0	9	3	12	6	1	9	4	13	7	2	10	5	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	11	5	0	8	3	11	6	1	9	4	12	7	1	10	4	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	10	5	13	8	3	11	6	0	9	3	12	6	1	10
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	9	4	12	7	1	10	5	13	8	2	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
8	3	11	6	0	9	3	12	7	1	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	0	9	3	12	6	1	9	4	12	7	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	13	7	2	11	5	0	8	3	11	6	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5

(b) H

P		P		P		Q			P		P		P		Q		
0	9	3	12	6	1	9	4	13	7	2	10	5	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	11	5	0	8	3	11	6	1	9	4	12	7	1	10	4	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	10	5	13	8	3	11	6	0	9	3	12	6	1	10
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	9	4	12	7	1	10	5	13	8	2	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
8	3	11	6	0	9	3	12	7	1	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	0	9	3	12	6	1	9	4	12	7	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	13	7	2	11	5	0	8	3	11	6	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5
0	9	3	12	6	1	9	4	13	7	2	10	5	13	8	2	11	6
1	10	4	13	7	2	10	5	0	8	3	11	6	0	9	3	12	7
2	11	5	0	8	3	11	6	1	9	4	12	7	1	10	4	13	8
3	12	6	1	9	4	12	7	2	10	5	13	8	2	11	5	0	9
4	13	7	2	10	5	13	8	3	11	6	0	9	3	12	6	1	10
5	0	8	3	11	6	0	9	4	12	7	1	10	4	13	7	2	11
6	1	9	4	12	7	1	10	5	13	8	2	11	5	0	8	3	12
7	2	10	5	13	8	2	11	6	0	9	3	12	6	1	9	4	13
8	3	11	6	0	9	3	12	7	1	10	4	13	7	2	10	5	0
9	4	12	7	1	10	4	13	8	2	11	5	0	8	3	11	6	1
10	5	13	8	2	11	5	0	9	3	12	6	1	9	4	12	7	2
11	6	0	9	3	12	6	1	10	4	13	7	2	10	5	13	8	3
12	7	1	10	4	13	7	2	11	5	0	8	3	11	6	0	9	4
13	8	2	11	5	0	8	3	12	6	1	9	4	12	7	1	10	5

Fig. 12. An example of Construction 4.1.

2. For $i = 2$ to m and for $j = 1$ to $p + q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L$.

3. Let s_{y_1}, s_{y_2}, $\cdots$, s_{y_q} be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$, and each s_{y_i} ($1 \leq i \leq q$) is the last element of a 'Q' in the offset sequence S.

For $i = 1$ to q, let $a_{y_i} = a_{y_i - 1} + L$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

□

Like Construction 4.1, the zigzag row constructed by Construction 4.2 also has a regular (and similar) structure.

Theorem 9: The zigzag rows constructed by Construction 4.1 and Construction 4.2 follow all the three rules — Rule 1, Rule 2 and Rule 3.

The above theorem can be proved with straightforward verification. So we skip its proof.

C. *Optimal Interleaving When t is Odd*

In this subsection, we prove that when t is odd, for a torus whose size is large enough in both dimensions, its t-interleaving number is at most one more than the sphere packing lower bound, $|S_t|$. We also present the corresponding optimal t-interleaving construction.

Lemma 6: In Equation Set (1) (the equation set in Construction 3.1), let the values of t, m and l_2 be fixed. Let '$p = p_0, q = q_0$' be a solution that satisfies the Equation Set (1). Then, another solution '$p = p_1, q = q_1$' also satisfies the Equation Set (1) if and only if there exists an integer c such that $p_1 = p_0 + c(m+1)(2m^2+2m+2) \geq 0$ and $q_1 = q_0 - cm(2m^2+2m+2) \geq 0$.

Proof: We can easily prove that "'$p = p_1, q = q_1$' is a solution that satisfies the Equation Set (1) if $p_1 = p_0 + c(m+1)(2m^2+2m+2) \geq 0$ and $q_1 = q_0 - cm(2m^2+2m+2) \geq 0$ for some integer c", by plugging '$p = p_1, q = q_1$' into the Equation Set (1). Now let's prove the other direction.

Assume '$p = p_1, q = q_1$' is a solution that satisfies the Equation Set (1). Let $x = p_1 - p_0$ and $y = q_1 - q_0$. By the first equation in Equation Set (1), $p_1 m + q_1(m+1) = l_2 = p_0 m + q_0(m+1)$ — therefore $(p_1 - p_0)m = -(q_1 - q_0)(m+1)$, which is $xm = -y(m+1)$. So x is a multiple of $m+1$ and y is a multiple of m. So there exists an integer a such that $x = a(m+1)$ and $y = -am$.

Now let us look at the second equation in Equation Set (1), $p_1(2m^2+m+1) + q_1(2m^2+3m+2) \equiv 0 \bmod (2m^2+2m+2)$. Note that $2m^2+m+1 \equiv -(m+1) \bmod (2m^2+2m+2)$ and $2m^2+3m+2 \equiv m \bmod (2m^2+2m+2)$. So $-p_1(m+1) + q_1 m \equiv 0 \bmod (2m^2+2m+2)$. Since $p_1 = p_0 + x = p_0 + a(m+1)$ and $q_1 = q_0 + y = q_0 - am$, we get $-[p_0 + a(m+1)](m+1) + (q_0 - am)m \equiv [-p_0(m+1) + q_0 m] - [a(m+1)^2 + am^2] \equiv -a(2m^2+2m+1) \equiv 0 \bmod (2m^2+2m+2)$. Since $2m^2+2m+1$ and $2m^2+2m+2$ must be relatively prime, we get $2m^2+2m+2|a$. So there exist an integer c such that $a = c(2m^2+2m+2)$. Then $p_1 = p_0 + x = p_0 + a(m+1) = p_0 + c(m+1)(2m^2+2m+2) \geq 0$ and $q_1 = q_0 + y = q_0 - am = q_0 - cm(2m^2+2m+2) \geq 0$.(The two inequalities come from the last condition in Equation Set (1).) That completes the proof of the other direction of this lemma.

□

Lemma 7: In Equation Set (1) (the equation set in Construction 3.1), let the values of t, m and l_2 be fixed. Let $\Delta_P = (m+1)(2m^2+2m+2)$ and $\Delta_Q = m(2m^2+2m+2)$. If there exists a solution of p and q that satisfies the Equation Set (1), then there exists a solution '$p = p^*, q = q^*$' that satisfies not only the Equation Set (1) but also one of the following two inequalities:

$$\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \leq p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2} \tag{3}$$

$$\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \leq p^* < q^* \leq \frac{l_2}{2m+1} + \frac{\Delta_Q}{2} \tag{4}$$

Proof: Assume there is a solution '$p = p_0, q = q_0$' that satisfies Equation Set (1). Trivially, either $p_0 \geq q_0$ or $p_0 < q_0$. Firstly, let us assume that $p_0 \geq q_0$. If $p_0 \geq \frac{l_2}{2m+1} + \Delta_P$, then $q_0 = \frac{l_2 - p_0 m}{m+1} \leq \frac{l_2 - [l_2/(2m+1)+\Delta_P]m}{m+1} = \frac{l_2 - [l_2/(2m+1)+(m+1)(2m^2+2m+2)]m}{m+1} = \frac{l_2}{2m+1} - \Delta_Q$ (and vice versa) — so then by Lemma 6, '$p = p_0 - \Delta_P, q = q_0 + \Delta_Q$' is also a solution to Equation Set (1), and what's more, $p_0 - \Delta_P \geq \frac{l_1}{2m+1} \geq q_0 + \Delta_Q$. Based on the above observation, we can see that there must exist a solution '$p = p_1, q = q_1$' such that $\frac{l_2}{2m+1} - \Delta_Q < q_1 \leq p_1 < \frac{l_2}{2m+1} + \Delta_P$. If $p_1 < \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$, then $q_1 > \frac{l_2}{2m+1} - \frac{\Delta_Q}{2}$ — then we can simply let $p^* = p_1$ and let $q^* = q_1$. If $p_1 \geq \frac{l_2}{2m+1} + \frac{\Delta_P}{2}$, then $q_1 \leq \frac{l_2}{2m+1} - \frac{\Delta_Q}{2}$ — then we will let $p^* = p_1 - \Delta_P$ and let $q^* = q_1 + \Delta_Q$, in which case we will have $\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \leq p^* < \frac{l_2}{2m+1} < q^* \leq \frac{l_2}{2m+1} + \frac{\Delta_Q}{2}$. So when $p_0 \geq q_0$, this lemma holds. The case that '$p_0 < q_0$' can be analyzed similarly.

□

Theorem 10: Let t be a positive odd integer. Let $m = \frac{t-1}{2}$. Define A as

$$\begin{aligned} A \quad = \quad \max\{ \quad & (\lceil \tfrac{l_2+(m+1)(2m+1)(m^2+m+1)}{l_2-m(2m+1)(m^2+m+1)} \rceil + 1)m^2 + 2m + 1, \\ & (\lceil \tfrac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)} \rceil + 1)m^2 + m + 2 - \lceil \tfrac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)} \rceil \} \end{aligned}$$

. Then when

$$l_2 \geq (m+1)(2m+1)(m^2+m+1) + 1$$

and

$$l_1 \geq (2m^2+2m+1)\left(\lceil \frac{A}{2m^2+2m+2} \rceil (2m^2+2m+2) - 2 \right)$$

, an $l_1 \times l_2$ (or equivalently, $l_2 \times l_1$) torus' t-interleaving number is either $|S_t|$ or $|S_t|+1$.

Proof: This theorem is trivially correct when $t=1$. When $t=3$, by the result of Appendix I (Theorem 13), we can also easily verify that this theorem is correct. So in the following analysis, we assume that $t>3$.

Let's first define a few variables for the ease of expression. Let $\Delta_P=(m+1)(2m^2+2m+2)$, $\Delta_Q=m(2m^2+2m+2)$, $B=\frac{l_2+(m+1)(2m+1)(m^2+m+1)}{l_2-m(2m+1)(m^2+m+1)}$, $C=\frac{l_2+m(2m+1)(m^2+m+1)}{l_2-(m+1)(2m+1)(m^2+m+1)}$, $D=(\lceil B\rceil+1)m^2+2m+1$, and $E=(\lceil C\rceil+1)m^2+m+2-\lceil C\rceil$. Then clearly $A=\max\{D,E\}$.

When $l_2\geq(m+1)(2m+1)(m^2+m+1)+1=(m+\frac{1}{2})(m+1)(2m^2+2m+2)+1>m(m+1)(2m^2+2m+2)=\lfloor\frac{t}{2}\rfloor(\lfloor\frac{t}{2}\rfloor+1)(|S_t|+1)$, by Theorem 6, there exists at least one solution of p and q that satisfies Equation Set (1). Then by Lemma 7, there exists a solution '$p=p^*,q=q^*$' to Equation Set (1) that satisfies either the condition $\frac{l_2}{2m+1}-\frac{\Delta_Q}{2}<q^*\leq p^*<\frac{l_2}{2m+1}+\frac{\Delta_P}{2}$ or the condition $\frac{l_2}{2m+1}-\frac{\Delta_P}{2}\leq p^*<q^*\leq\frac{l_2}{2m+1}+\frac{\Delta_Q}{2}$. We analyze the two cases below.

- Case 1: there is a solution '$p=p^*,q=q^*$' to Equation Set (1) that satisfies the condition $\frac{l_2}{2m+1}-\frac{\Delta_Q}{2}<q^*\leq p^*<\frac{l_2}{2m+1}+\frac{\Delta_P}{2}$. We use Construction 3.1 to t-interleave an $(|S_t|+1)\times l_2$ torus G_1. Note that when $l_2\geq(m+1)(2m+1)(m^2+m+1)+1$, $\frac{l_2}{2m+1}-\frac{\Delta_Q}{2}>0$, so $q^*>0$. Also note that $\frac{p^*}{q^*}<\frac{l_2/(2m+1)+\Delta_P/2}{l_2/(2m+1)-\Delta_Q/2}=B$, so $D\geq(\lceil\frac{p^*}{q^*}\rceil+1)m^2+2m+1$. Let G_2 be an $[\lceil\frac{D}{|S_t|+1}\rceil(|S_t|+1)]\times l_2$ torus got by tiling $\lceil\frac{D}{|S_t|+1}\rceil$ copies of G_1 vertically. We use Construction 4.1 to find a zigzag row in G_2; then by removing the zigzag row in G_2, we get a torus G_3 whose size is $[\lceil\frac{D}{|S_t|+1}\rceil(|S_t|+1)-1]\times l_2$. Clearly the number of rows in G_1, $|S_t|+1$, and the number of rows in G_3, $\lceil\frac{D}{|S_t|+1}\rceil(|S_t|+1)-1$, are relatively prime. So for any $l_0\times l_2$ torus G where $l_0\geq(|S_t|+1-1)(\lceil\frac{D}{|S_t|+1}\rceil(|S_t|+1)-1-1)=|S_t|(\lceil\frac{D}{|S_t|+1}\rceil(|S_t|+1)-2)$, it can be got by tiling copies of G_1 and G_3 vertically — and by Lemma 5, G is t-interleaved, with the t-interleaving degree of $|S_t|+1$.
- Case 2: there is a solution '$p=p^*,q=q^*$' to Equation Set (1) that satisfies the condition $\frac{l_2}{2m+1}-\frac{\Delta_P}{2}\leq p^*<q^*\leq\frac{l_2}{2m+1}+\frac{\Delta_Q}{2}$. We use Construction 3.1 to t-interleave an $(|S_t|+1)\times l_2$ torus G_1. Note that when $l_2\geq(m+1)(2m+1)(m^2+m+1)+1$, $\frac{l_2}{2m+1}-\frac{\Delta_P}{2}>0$, so $p^*>0$. Also note that $\frac{q^*}{p^*}\leq\frac{l_2/(2m+1)+\Delta_Q/2}{l_2/(2m+1)-\Delta_P/2}=C$, so $E\geq(\lceil\frac{q^*}{p^*}\rceil+1)m^2+m+(2-\lceil\frac{q^*}{p^*}\rceil)$. Let G_2 be an $[\lceil\frac{E}{|S_t|+1}\rceil(|S_t|+1)]\times l_2$ torus got by tiling $\lceil\frac{E}{|S_t|+1}\rceil$ copies of G_1 vertically. We use Construction 4.2 to find a zigzag row in G_2; then by removing the zigzag row in G_2, we get a torus G_3 whose size is $[\lceil\frac{E}{|S_t|+1}\rceil(|S_t|+1)-1]\times l_2$. Clearly the number of rows in G_1, $|S_t|+1$, and the number of rows in G_3, $\lceil\frac{E}{|S_t|+1}\rceil(|S_t|+1)-1$, are relatively prime. So for any $l_0\times l_2$ torus G where $l_0\geq(|S_t|+1-1)(\lceil\frac{E}{|S_t|+1}\rceil(|S_t|+1)-1-1)=|S_t|(\lceil\frac{E}{|S_t|+1}\rceil(|S_t|+1)-2)$, it can be got by tiling copies of G_1 and G_3 vertically — and by Lemma 5, G is t-interleaved, with the t-interleaving degree of $|S_t|+1$.

Now let G be an $l_1\times l_2$ torus where $l_2\geq(m+1)(2m+1)(m^2+m+1)+1$ and $l_1\geq(2m^2+2m+1)(\lceil\frac{A}{2m^2+2m+2}\rceil(2m^2+2m+2)-2)=|S_t|(\lceil\frac{\max\{D,E\}}{|S_t|+1}\rceil(|S_t|+1)-2)$. Based on the analysis for Case (1) and Case (2), we know that G's t-interleaving number is at most $|S_t|+1$. By the sphere packing lower bound, G's t-interleaving number is at least $|S_t|$. So G's t-interleaving number is either $|S_t|$ or $|S_t|+1$.
□

For easy reference, we show the method for optimally t-interleaving a large torus as a construction below. Note that the construction below is applicable only when $t\geq5$ (and by default, t is odd). When $t=1$, any torus can be t-interleaved with 1 integer in a trivial way. When $t=3$, the torus can be t-interleaved with the construction to be presented in Appendix I.

Construction 4.3: Optimal t-Interleaving on a Large Torus
Input: An odd integer t such that $t\geq5$. An integer m such that $m=\frac{t-1}{2}$. An $l_1\times l_2$ torus, where

$$l_2\geq(m+1)(2m+1)(m^2+m+1)+1$$

and

$$l_1\geq(2m^2+2m+1)\left(\lceil\frac{A}{2m^2+2m+2}\rceil(2m^2+2m+2)-2\right)$$

. (The parameter A is as defined in Theorem 10.)
Output: An optimal t-interleaving on the $l_1\times l_2$ torus.

Construction:

1. If both l_1 and l_2 are multiples of $|S_t|$, then the $l_1 \times l_2$ torus' t-interleaving number is $|S_t|$. In this case, we use Construction 2.2 to t-interleave the $l_1 \times l_2$ torus with $|S_t|$ distinct integers.

2. If either l_1 or l_2 is not a multiple of $|S_t|$, then the $l_1 \times l_2$ torus' t-interleaving number is $|S_t|+1$. In this case, we t-interleave the torus with $|S_t|+1$ integers in the following way: firstly, we t-interleave an $(|S_t|+1) \times l_2$ torus, B, by using Construction 3.1 (note that $|S_t|+1 = 2m^2+2m+2$); secondly, let H be an $[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1)] \times l_2$ torus which is got by tiling $\lceil \frac{A}{|S_t|+1} \rceil$ copies of B vertically, and use Construction 4.1 or Construction 4.2 (depending on which is applicable) to find a zigzag row in H; thirdly, remove the zigzag row in H to get a $[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1)-1] \times l_2$ torus T; finally, find non-negative integers x and y such that $l_1 = x(|S_t|+1) + y[\lceil \frac{A}{|S_t|+1} \rceil (|S_t|+1)-1]$, and get an $l_1 \times l_2$ torus by tiling x copies of B and y copies of T vertically. The resulting interleaving on the $l_1 \times l_2$ torus is a t-interleaving.

□

D. *Optimal Interleaving When t Is Even*

When t is even, the optimal t-interleaving on large tori can be analyzed in a very similar way as in the case of odd t. The main result for even t is shown in the following theorem. For succinctness, we leave the major steps and intermediate results of the corresponding analysis in Appendix II.

Theorem 11: Let t be a positive even integer. Let $m = \frac{t}{2}$. Define A as

$$\begin{aligned} A \quad = \quad \max\{ \quad & (\lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \rceil + 1)m^2 + (3 - \lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \rceil)m - 3, \\ & (\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil + 1)m^2 + (3 - \lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil)m - 1 \\ & -2\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil \} \end{aligned}$$

. Then when

$$l_2 > \frac{(m+1)(2m+1)(2m^2+1)}{2}$$

and

$$l_1 \geq 2m^2 \left(\lceil \frac{A}{2m^2+1} \rceil (2m^2+1) - 2 \right)$$

, an $l_1 \times l_2$ (or equivalently, $l_2 \times l_1$) torus' t-interleaving number is either $|S_t|$ or $|S_t|+1$.

V. General Bounds on Interleaving Numbers

We have shown that for a torus whose size is large enough in both dimensions (Theorem 10 and Theorem 11), its t-interleaving number is at most $|S_t|+1$. If the requirement on the torus' size is loosened to some extent (Theorem 8), then its t-interleaving number is at most $|S_t|+2$. Does that mean for a torus of any size, its t-interleaving number is always at most $|S_t|$ plus a small constant? The answer is no. The following theorem shows bounds on t-interleaving numbers.

Theorem 12: (1) The t-interleaving numbers of two-dimensional tori are $|S_t| + O(t^2)$ in general. And that upper bound is tight, even if the following restriction is enforced on the tori — the number of rows or the number of columns of the torus approaches infinity. (2) When both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.

Proof: (1) Firstly, let's show that the t-interleaving numbers of two-dimensional tori are $|S_t| + O(t^2)$ in general. Let G be an $l_1 \times l_2$ torus. First we assume that t is even and $l_1 \geq t$, $l_2 \geq t$. Let $K_1 = \lfloor \frac{l_1}{t} \rfloor$, $K_2 = \lfloor \frac{l_2}{t} \rfloor$. We see G as being tiled by small blocks in the way shown in Fig. 13, where the blocks are labelled by 'A' or 'B'. (Note that two blocks both labelled as 'A' are not necessary of the same size. And two blocks both labelled as 'B' are not necessary of the same size, either.) For every block labelled as 'A' (respectively, 'B'), the four blocks around it (to its left, right, up and down) are all labelled as 'B' (respectively, 'A'). Each block consists of either $\lceil \frac{l_1}{2K_1} \rceil$ or $\lfloor \frac{l_1}{2K_1} \rfloor$ rows, and either $\lceil \frac{l_2}{2K_2} \rceil$ or $\lfloor \frac{l_2}{2K_2} \rfloor$ columns. (Note that

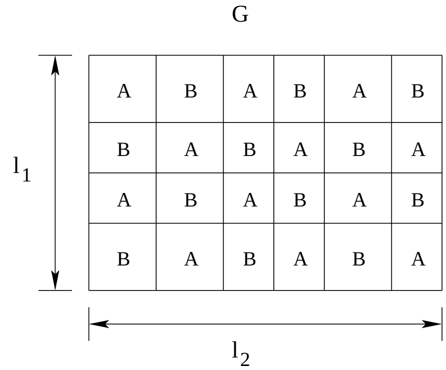


Fig. 13. See G as being tiled by small blocks.

$\lceil \frac{l_1}{2K_1} \rceil = \lceil \frac{K_1 t + (l_1 \bmod t)}{2K_1} \rceil = \frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil$, $\lfloor \frac{l_1}{2K_1} \rfloor = \frac{t}{2} + \lfloor \frac{l_1 \bmod t}{2K_1} \rfloor$, $\lceil \frac{l_2}{2K_2} \rceil = \frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil$, $\lfloor \frac{l_2}{2K_2} \rfloor = \frac{t}{2} + \lfloor \frac{l_2 \bmod t}{2K_2} \rfloor$.) We see each block as a torus of its corresponding size. (So for a block whose size is $\alpha \times \beta$, it vertices are denoted by (i, j) for $i = 0, 1, \cdots, \alpha - 1$ and $j = 0, 1, \cdots, \beta$, in the same way a torus' vertices are normally denoted.) Now we interleave all the blocks following these two rules: (i) only integers in the set $\{1, 2, \cdots, \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil\}$ are used to interleave any block 'A', and only integers in the set $\{\lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil + 1, \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil + 2, \cdots, 2 \cdot \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil\}$ are used to interleave any block 'B'; (ii) for all the blocks labelled by 'A' (respectively, 'B') and for any i and j, the vertices denoted by (i, j) in them (provided they exist) are all labelled by the same integer. It is very easy to see that G is t-interleaved in this way, using $2 \cdot \lceil \frac{l_1}{2K_1} \rceil \cdot \lceil \frac{l_2}{2K_2} \rceil = 2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil) \le 2(\frac{t}{2} + \lceil \frac{t-1}{2} \rceil)(\frac{t}{2} + \lceil \frac{t-1}{2} \rceil) = 2t^2 = |S_t| + \frac{3}{2}t^2$ distinct integers. So G's t-interleaving number is $|S_t| + O(t^2)$.

Now we assume t is even, and $l_1 < t$ or $l_2 < t$. Without loss of generality, let's say $l_1 < t$. Then we see G as being tiled horizontally by smaller tori A_1, A_2, $\cdots$, A_n, where each A_i — for $i = 1, 2, \cdots, n-1$ — is an $l_1 \times t$ torus, and A_n is an $l_1 \times (l_2 \bmod t)$ torus. We interleave A_1, A_2, $\cdots$, A_{n-1} in exactly the same way, and assign $l_1 \times t$ distinct integers to each of them. We interleave A_n with a disjoint set of $l_1 \times (l_2 \bmod t)$ integers. Clearly G is t-interleaved in this way, using $l_1 \cdot t + l_1 \cdot (l_2 \bmod t) = |S_t| + O(t^2)$ distinct integers. So again, G's t-interleaving number is $|S_t| + O(t^2)$.

Finally we assume t is odd. We can $(t+1)$-interleave G using $|S_{t+1}| + O((t+1)^2) = \frac{(t+1)^2}{2} + O((t+1)^2) = \frac{t^2+1}{2} + O(t^2) = |S_t| + O(t^2)$ distinct integers. $t+1$ is even, and a $(t+1)$-interleaving is also a t-interleaving. So G's t-interleaving number is still $|S_t| + O(t^2)$.

Now let's show that the above bound on t-interleaving numbers, $|S_t| + O(t^2)$, is tight, no matter if t is even or odd. Consider an $l_1 \times l_2$ torus where l_1 is the largest even integer that is no greater than $\lfloor \frac{3}{2}t \rfloor$, and l_2 is any integer greater than or equal to $\lfloor \frac{3}{4}t \rfloor$. We are firstly going to show that a t-interleaving can place an integer at most twice in any $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus.

Assume a t-interleaving places an integer on three vertices in $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus. Without loss of generality, let's say those three vertices are $(0, 0)$, (i_1, j_1) and (i_2, j_2), where $0 \le j_1 \le \lfloor \frac{3}{4}t \rfloor - 1$ and $0 \le j_2 \le \lfloor \frac{3}{4}t \rfloor - 1$. Since the interleaving is a t-interleaving, the Lee distance between any two of those three vertices is at least t. Let $a = \frac{l_1}{2}$ and $b = \lfloor \frac{3}{4}t \rfloor - 1$. It is not difficult to see that the Lee distance between (i_1, j_1) and (a, b) is at most $\min\{(a - i_1) \bmod l_1, (i_1 - a) \bmod l_1\} + (b - j_1) = \frac{l_1}{2} - \min\{(0 - i_1) \bmod l_1, (i_1 - 0) \bmod l_1\} + (b - j_1) = \frac{l_1}{2} + b - [\min\{(0 - i_1) \bmod l_1, (i_1 - 0) \bmod l_1\} + j_1]$. Since the Lee distance between $(0, 0)$ and (i_1, j_1) is at most $\min\{(0 - i_1) \bmod l_1, (i_1 - 0) \bmod l_1\} + j_1$, we know that $\min\{(0 - i_1) \bmod l_1, (i_1 - 0) \bmod l_1\} + j_1 \ge t$. Therefore the Lee distance between (i_1, j_1) and (a, b) is at most $\frac{l_1}{2} + b - t \le \lfloor \frac{3}{2}t \rfloor / 2 + \lfloor \frac{3}{4}t \rfloor - 1 - t < \frac{t}{2}$. Similarly, the Lee distance between (i_2, j_2) and (a, b) is also less than $\frac{t}{2}$. Therefore the Lee distance between (i_1, j_1) and (i_2, j_2) is less than t, which is a contradiction. So a t-interleaving can place every integer on at most two vertices in $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the torus.

Any $\lfloor \frac{3}{4}t \rfloor$ consecutive columns of the $l_1 \times l_2$ torus contain $l_1 \times \lfloor \frac{3}{4}t \rfloor \geq (\frac{3}{2}t - 2) \times (\frac{3}{4}t - 1) = \frac{9}{8}t^2 - 3t + 2$ vertices, where each integer is placed at most twice by a t-interleaving. Therefore the t-interleaving number of the torus is at least $\frac{\frac{9}{8}t^2 - 3t + 2}{2} = \frac{9}{16}t^2 - \frac{3}{2}t + 1 = \frac{t^2+1}{2} + \frac{1}{16}t^2 - \frac{3}{2}t + \frac{1}{2} \geq |S_t| + \frac{1}{16}t^2 - \frac{3}{2}t + \frac{1}{2} = |S_t| + \Theta(t^2)$, which matches the upper bound $|S_t| + O(t^2)$. Since here l_2 can be *any* integer that is no less than $\lfloor \frac{3}{4}t \rfloor$, the upper bound is tight even if the number of columns (or equivalently, the number of rows) of the torus approaches infinity. The first part of this theorem has been proved by now.

(2) Let's prove the second part of this theorem. In the previous part of this proof, a method for t-interleaving an $l_1 \times l_2$ torus has been proposed for the case 't is even and $l_1 \geq t$, $l_2 \geq t$'. That method uses $2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil)$ distinct integers. (Note that $K_1 = \lfloor \frac{l_1}{t} \rfloor$ and $K_2 = \lfloor \frac{l_2}{t} \rfloor$.) When both l_1 and l_2 are of the order $\Omega(t^2)$, both K_1 and K_2 are of the order of $\Omega(t)$ — and then $2(\frac{t}{2} + \lceil \frac{l_1 \bmod t}{2K_1} \rceil)(\frac{t}{2} + \lceil \frac{l_2 \bmod t}{2K_2} \rceil) = 2(\frac{t}{2} + O(1))(\frac{t}{2} + O(1)) = \frac{t^2}{2} + O(t) = |S_t| + O(t)$. When t is odd, we can t-interleave an $l_1 \times l_2$ torus, where $l_1 = \Omega(t^2) = \Omega((t+1)^2)$ and $l_2 = \Omega(t^2) = \Omega((t+1)^2)$, by $(t+1)$-interleaving it using $|S_{t+1}| + O(t+1) = \frac{(t+1)^2}{2} + O(t) = \frac{t^2+1}{2} + O(t) = |S_t| + O(t)$ distinct integers. So no matter if t is even or odd, when both l_1 and l_2 are of the order $\Omega(t^2)$, the t-interleaving number of an $l_1 \times l_2$ torus is $|S_t| + O(t)$.

□

VI. Discussions

In this paper, we study the t-interleaving problem for two-dimensional tori. It has applications in both distributed data storage and burst error correction. This is the first time that the t-interleaving problem is studied for graphs with modular structures, and consequently, novel interleaving methods different from traditional techniques (e.g., the widely used lattice-interleaver schemes in early works [11], [13], [25]) are developed for optimal t-interleaving. The necessary and sufficient condition for tori that can be perfectly t-interleaved is proven, and the corresponding perfect t-interleaving construction is presented, based on the method of sphere packing. The most important contribution of this paper is to prove that for tori whose sizes are large in both dimensions, which constitute by far the majority of all existing cases, their t-interleaving numbers are at most one more than the sphere packing lower bound. Optimal t-interleaving constructions for such tori are presented, based on the method of removing-a-zigzag-row and tori-tiling. Then, some additional bounds on the t-interleaving numbers are shown. Those results together give a general characterization of the t-interleaving problem for two-dimensional tori.

The importance of the t-interleaving method based on removing-a-zigzag-row and tori-tiling is not limited to the results in Theorem 10 and Theorem 11. Those two theorems should be seen as a lower bound for the performance of the t-interleaving method. By analyzing the performance of the corresponding t-interleaving constructions more carefully, and furthermore, by keeping the main idea of the t-interleaving method but tuning its specific parameters on a case-by-case basis, we can improve the bounds derived in Theorem 10 and Theorem 11. The content of Appendix I can serve as an example in this aspect. What's more, the t-interleaving method can be used to optimally t-interleave some tori whose sizes do not fall within the derived bounds.

We are interested in studying the t-interleaving problem for higher-dimensional tori, as well as finding more t-interleaving methods. Those remain as our future research.

Appendix I

The optimal t-interleaving construction for odd t, Construction 4.3, if applicable only when $t \geq 5$. In this appendix, we present the optimal t-interleaving construction when $t = 3$, thus completing the result for t-interleaving on large tori while t being odd. We also use this case, $t = 3$, as an example to show how previous results can be improved if the t-interleaving problem is analyzed case by case and more carefully.

We will show that when $l_1 \geq 20$ and $l_2 \geq 15$ (or equivalently, when $l_1 \geq 15$ and $l_2 \geq 20$), an $l_1 \times l_2$ torus' 3-interleaving number is either 5 or 6. (Note that $|S_3| = 5$.) Below we present an construction that can optimally 3-interleaves any $l_1 \times l_2$ torus where $l_1 \geq 20$ and $l_2 \geq 15$, except when $l_2 = 19$.

Construction 4.4: Optimally 3-Interleave an $l_1 \times l_2$ torus, where $l_1 \geq 20$, $l_2 \geq 15$, and $l_2 \neq 19$.

1. If both l_1 and l_2 are multiples of 5, then the $l_1 \times l_2$ torus' 3-interleaving number is $|S_t| = 5$. In this case, 3-interleave the $l_1 \times l_2$ torus with 5 integers by using Construction 2.2.

If l_1 or l_2 is not a multiple of 5, then use the following three steps to 3-interleave the $l_1 \times l_2$ torus with 6 integers.

2. Find non-negative integers x_1 and x_2 such that $l_1 = 5x_1 + 6x_2$. Find non-negative integers y_1, y_2 and y_3 such that $l_2 = 5y_1 + 8y_2 + 12y_3$.

3. There are six tori shown in Fig. 14(a)— an 5×5 torus 'A', an 5×8 torus 'B', an 5×12 torus 'C', an 6×5 torus 'A'', an 6×8 torus 'B'' and an 6×12 torus 'C''.

Get a $5 \times l_2$ torus M_1 by tiling horizontally y_1 copies of 'A', y_2 copies of 'B' and y_3 copies of 'C' (whose order can be arbitrary).

Get a $6 \times l_2$ torus M_2 by tiling horizontally y_1 copies of 'A'', y_2 copies of 'B'' and y_3 copies of 'C'', whose order needs to satisfy this rule: for $i = 1$ to $y_1 + y_2 + y_3$, if the i-th module-torus in M_1 is an 'A' (respectively, a 'B' or a 'C'), then the i-th module in M_2 is an 'A'' (respectively, a 'B'' or a 'C'').

4. Get an $l_1 \times l_2$ torus by tiling x_1 copies of M_1 and x_2 copies of M_2 vertically (whose order can be arbitrary). The interleaving on the $l_1 \times l_2$ torus is a 3-interleaving.

□

Example: We use Construction 4.4 to 3-interleave an $l_1 \times l_2$ torus where $l_1 = 11$ and $l_2 = 25$. l_1 is not a multiple of $|S_t|$, so the torus' 3-interleaving number is greater than 5. Since $l_1 = 5+6$ and $l_2 = 5+8+12$, the variables in Construction 4.4 can be set as follows: $x_1 = 1$, $x_2 = 1$, $y_1 = 1$, $y_2 = 1$ and $y_3 = 1$. And we can let the torus M_1 have the form of $[ABC]$, and let the torus M_2 have the form of $[A'B'C']$. We then tile M_1 and M_2 to get the $l_1 \times l_2$ torus, which is of the form $\begin{bmatrix} A & B & C \\ A' & B' & C' \end{bmatrix}$. This 3-interleaved torus is shown in Fig. 14(b). The interleaving used $6 = |S_3| + 1$ integers.

Clearly, since $25 = 5 \times 5 + 8 \times 0 + 12 \times 0$, another choice to tile the 11×25 torus is $\begin{bmatrix} A & A & A & A & A \\ A' & A' & A' & A' & A' \end{bmatrix}$.

□

Construction 4.4 constructs a 3-interleaved $l_1 \times l_2$ torus by tiling copies of 6 module-tori — the 6 tori shown in Fig. 14(a). It can be readily verified that when those 6 tori are tiled following the rule in Construction 4.4, the resulting interleaving on the $l_1 \times l_2$ torus is indeed a 3-interleaving. There are only a limited number of cases to analyze for the verification, so we skip the details. We comment that Construction 4.4 does not work for the case $l_2 = 19$, because 19 cannot be written as a linear combination of 5, 8 and 12 with non-negative coefficients — therefore an $l_1 \times 19$ torus cannot be got by tiling the module-tori. We present the construction for the case $l_2 = 19$ below.

Construction 4.5: Optimally 3-Interleave an $l_1 \times 19$ torus, where $l_1 \geq 20$.

Construction: Find non-negative integers x_1 and x_2 such that $l_1 = 5x_1 + 6x_2$. There are 2 tori shown in Fig. 15 — a 5×19 torus F and a 6×19 torus F'. Get an $l_1 \times 19$ torus by tiling x_1 copies of F and x_2 copies of F' vertically (whose order can be arbitrary). The resulting interleaving on the $l_1 \times 19$ torus is a 3-interleaving.

□

The correctness of Construction 4.5 can be easily verified, so we skip the details. Based on the previous two constructions, we readily get the following conclusion for 3-interleaving.

Theorem 13: When $l_1 \geq 20$ and $l_2 \geq 15$, or when $l_1 \geq 15$ and $l_2 \geq 20$, an $l_1 \times l_2$ torus' 3-interleaving number is either $|S_3|$ or $|S_3| + 1$.

(a) Modules

A

0	2	4	1	3
1	3	0	2	5
2	5	1	3	0
4	0	2	5	1
5	1	3	0	2

B

0	2	4	0	3	5	1	3
1	3	5	2	4	0	2	5
2	4	1	3	5	1	4	0
3	0	2	4	0	3	5	1
5	1	3	5	2	4	0	2

C

0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	4	0	3	5	2	4
2	5	1	4	0	3	5	2	4	1	3	0
4	0	3	5	2	4	1	3	0	2	5	1
5	1	4	0	3	5	2	4	1	3	0	2

A'

0	2	4	1	3
1	3	5	2	4
2	4	0	3	5
3	5	1	4	0
4	0	2	5	1
5	1	3	0	2

B'

0	2	4	0	3	5	1	3
1	3	5	1	4	0	2	4
2	4	0	2	5	1	3	5
3	5	1	3	0	2	4	0
4	0	2	4	1	3	5	1
5	1	3	5	2	4	0	2

C'

0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	4	0	3	5	2	4
2	4	1	3	0	2	5	1	4	0	3	5
3	5	2	4	1	3	0	2	5	1	4	0
4	0	3	5	2	4	1	3	0	2	5	1
5	1	4	0	3	5	2	4	1	3	0	2

(b) Tiling of modules

0	2	4	1	3	0	2	4	0	3	5	1	3	0	2	5	1	4	0	3	5	2	4	1	3
1	3	0	2	5	1	3	5	2	4	0	2	5	1	3	0	2	5	1	4	0	3	5	2	4
2	5	1	3	0	2	4	1	3	5	1	4	0	2	5	1	4	0	3	5	2	4	1	3	0
4	0	2	5	1	3	0	2	4	0	3	5	1	4	0	3	5	2	4	1	3	0	2	5	1
5	1	3	0	2	5	1	3	5	2	4	0	2	5	1	4	0	3	5	2	4	1	3	0	2
0	2	4	1	3	0	2	4	0	3	5	1	3	0	2	5	1	4	0	3	5	2	4	1	3
1	3	5	2	4	1	3	5	1	4	0	2	4	1	3	0	2	5	1	4	0	3	5	2	4
2	4	0	3	5	2	4	0	2	5	1	3	5	2	4	1	3	0	2	5	1	4	0	3	5
3	5	1	4	0	3	5	1	3	0	2	4	0	3	5	2	4	1	3	0	2	5	1	4	0
4	0	2	5	1	4	0	2	4	1	3	5	1	4	0	3	5	2	4	1	3	0	2	5	1
5	1	3	0	2	5	1	3	5	2	4	0	2	5	1	4	0	3	5	2	4	1	3	0	2

Fig. 14. Using modules for 3-interleaving. (a) The 6 modules; (b) Tiling the modules.

F

0	2	4	1	3	5	1	3	0	2	4	0	2	5	1	3	5	1	4
1	3	0	2	4	0	2	5	1	3	5	1	4	0	2	4	0	3	5
2	5	1	3	5	1	4	0	2	4	0	3	5	1	3	5	2	4	0
4	0	2	4	0	3	5	1	3	5	2	4	0	2	4	1	3	5	1
5	1	3	5	2	4	0	2	4	1	3	5	1	3	0	2	4	0	3

F'

0	2	4	0	3	5	1	3	5	2	4	0	2	4	1	3	5	1	4
1	3	5	1	4	0	2	4	0	3	5	1	3	5	2	4	0	2	5
2	4	0	2	5	1	3	5	1	4	0	2	4	0	3	5	1	3	0
3	5	1	3	0	2	4	0	2	5	1	3	5	1	4	0	2	4	1
4	0	2	4	1	3	5	1	3	0	2	4	0	2	5	1	3	5	2
5	1	3	5	2	4	0	2	4	1	3	5	1	3	0	2	4	0	3

Fig. 15. Two modules used for 3-Interleaving an $l_1 \times 19$ torus, where $l_1 \geq 20$.

We comment that the result we got here is comparatively better than the result derived in Section IV. (For example, if Theorem 10 is applied for the case $t = 3$, then the bound for l_2 would be 19. However here our bound for l_2 is 15.) However, we should notice that the t-interleaving method used here is the same as the method used for $t > 3$ *per se*. (We can see that the module-tori 'A', 'B', 'C' in Fig. 14(a) and 'F' in Fig. 15 are got by removing a zigzag row from 'A'', 'B'', 'C'' and 'F''. The zigzag rows are shown in circles in those two figures. Both the interleaving method here and the method in Section IV are based on torus tiling.) The improvement is made by better tuning of construction parameters and more careful analysis of the bounds. The construction used for $t = 3$ does not follow all the requirements used in Section IV. (For example, the zigzag row in Fig. 15 does not follow Rule 3.) In Section IV, while endeavoring to optimally tune all the parameters, we also need to ensure that the construction will work for all the cases of $t > 3$. If the interleaving problem is analyzed case by case (specifically, for each value of t, l_1 and l_2), the interleaving construction has room for further optimization.

APPENDIX II

In this appendix, we show how to optimally t-interleave large tori when t is even. The process is similar to the case where t is odd, differing only in details. For this reason, we just present a succinct description of the process and results. This appendix's content is parallel to that of the first three subsections of Section IV, so comparative reading should help the understanding greatly.

We assume t is even throughout the remainder of this appendix. The definitions of '*a zigzag row*' and '*removing a zigzag row*' are the same as in Definition 4.1 and 4.2.

Let B be an $l_0 \times l_2$ torus which is t-interleaved by Construction 3.1 utilizing the offset sequence $S =$ '$s_0, s_1, \cdots, s_{l_2-1}$'. Let H be an $l_1 \times l_2$ torus got by tiling several copies of B vertically. Let $m = \frac{t}{2}$. There are four rules to follow for devising a zigzag row — denoted by $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$ — in H:

- Rule 1: For any j such that $0 \le j \le l_2 - 1$, if the integers $s_j, s_{(j+1) \bmod l_2}, \cdots, s_{(j+m-1) \bmod l_2}$ do not all equal $t - 1$, then $a_j \ge a_{(j+m) \bmod l_2} + m - 1$.
- Rule 2: For any j such that $0 \le j \le l_2 - 1$, if exactly one of the integers $s_j, s_{(j+1) \bmod l_2}, \cdots, s_{(j+m) \bmod l_2}$ equals t, then $a_j \le a_{(j+m+1) \bmod l_2} - (m - 2)$.
- Rule 3: For any j such that $0 \le j \le l_2 - 1$, if $s_j = t - 1$, then $a_j \le a_{(j+1) \bmod l_2} - (2m - 2)$.
- Rule 4: For any j such that $0 \le j \le l_2 - 1$, $2m - 2 \le a_j \le l_1 - 1 - (2m - 2)$.

Lemma 8: Let B be a torus t-interleaved by Construction 3.1. Let H be a torus got by tiling copies of B vertically, and let T be a torus got by removing a zigzag row in H, where the zigzag row in H follows the four rules — Rule 1, Rule 2, Rule 3 and Rule 4. Let G be a torus got by tiling copies of B and T vertically. Then, both T and G are t-interleaved.

Now we present two constructions for finding a zigzag row, which are the counterparts of Construction 4.1 and 4.2. Let B be an $l_0 \times l_2$ torus which is t-interleaved by Construction 3.1 utilizing the offset sequence $S =$ '$s_0, s_1, \cdots, s_{l_2-1}$'. Let H be an $l_1 \times l_2$ torus got by tiling z copies of G vertically. We say the offset sequence S consists of p 'P's and q 'Q's, where $p > 0$ and $q > 0$. We require that in S, the 'P's and 'Q's are interleaved very evenly, and that S starts with a P and ends with a Q. Let $m = \frac{t}{2}$. Let $L = (2m - 2) + (m - 1)\lceil \frac{p}{q} \rceil$ if $p \ge q$, and let $L = (2m - 2) + (m - 2)\lceil \frac{q}{p} \rceil + 1$ if $p < q$. We require that $l_1 \ge (\lceil \frac{p}{q} \rceil + 1)m^2 + (3 - \lceil \frac{p}{q} \rceil)m - 3$ if $p \ge q$, and require that $l_1 \ge (\lceil \frac{q}{p} \rceil + 1)m^2 + (3 - \lceil \frac{q}{p} \rceil)m - (2\lceil \frac{q}{p} \rceil + 1)$ if $p < q$. Below we present two constructions for constructing a zigzag row, which is denoted by $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, in H, applicable respectively when $p \ge q$ and $p < q$.

Construction 4.6: Constructing a zigzag row in H, when t is even, $t > 2$, and $p \ge q > 0$

1. Let s_{x_1}, s_{x_2}, $\cdots$, $s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$, and each s_{x_i} ($1 \le i \le p + q$) is the first element of a 'P' or 'Q' in the offset sequence S.

   Let $a_{x_1} = L$. For $i = 2$ to $p + q$, if $s_{x_{i-1}}$ is the first element of a 'Q', let $a_{x_i} = L$.

For $i = 2$ to $p+q$, if $s_{x_{i-1}}$ is the first element of a 'P', then let $a_{x_i} = a_{x_{i-1}} - (m-1)$.

2. For $i = 2$ to m and for $j = 1$ to $p+q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L - m + 1$.

3. Let $s_{y_1}, s_{y_2}, \cdots, s_{y_q}$ be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$, and each s_{y_i} $(1 \le i \le q)$ is the last element of a 'Q' in the offset sequence S.

For $i = 1$ to q, $a_{y_i} = L + (m-1)(L-m+1) + (m-1)$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

□

Construction 4.7: Constructing a zigzag row in H, when t is even, $t > 2$, and $0 < p < q$

1. Let $s_{x_1}, s_{x_2}, \cdots, s_{x_{p+q}}$ be the integers such that $0 = x_1 < x_2 < \cdots < x_{p+q} = l_2 - m - 1$, and each s_{x_i} $(1 \le i \le p+q)$ is the first element of a 'P' or 'Q' in the offset sequence S.

Let $a_{x_1} = L$. For $i = 2$ to $p+q$, if s_{x_i} is the first element of a 'P', then let $a_{x_i} = L$; if $s_{x_{i-1}}$ is the first element of a 'P', then let $a_{x_i} = L - \lceil \frac{q}{p} \rceil (m-2) - 1$; otherwise, let $a_{x_i} = a_{x_{i-1}} + (m-2)$.

2. For $i = 2$ to m and for $j = 1$ to $p+q$, let $a_{x_j+i-1} = a_{x_j+i-2} + L - m + 1$.

3. Let $s_{y_1}, s_{y_2}, \cdots, s_{y_q}$ be the integers such that $y_1 < y_2 < \cdots < y_q = l_2 - 1$, and each s_{y_i} is the last element of a 'Q' in the offset sequence S.

For $i = 1$ to q, $a_{y_i} = a_{y_i - 1} + L - m + 1$.

Now we have fully determined the zigzag row, $\{(a_0, 0), (a_1, 1), \cdots, (a_{l_2-1}, l_2 - 1)\}$, in the torus H.

□

Theorem 14: The zigzag rows constructed by Construction 4.6 and Construction 4.7 follow all the four rules — Rule 1, Rule 2, Rule 3 and Rule 4.

Lemma 9: In Equation Set (2) (which is in Construction 3.1), let the values of t, m and l_2 be fixed. Let '$p = p_0, q = q_0$' be a solution that satisfies the Equation Set (2). Then, another solution '$p = p_1, q = q_1$' also satisfies the Equation Set (2) if and only if there exists an integer c such that $p_1 = p_0 + c(m+1)(2m^2+1) \ge 0$ and $q_1 = q_0 - cm(2m^2+1) \ge 0$.

Lemma 10: In Equation Set (2) (which is in Construction 3.1), let the values of t, m and l_2 be fixed. Let $\Delta_P = (m+1)(2m^2+1)$ and $\Delta_Q = m(2m^2+1)$. If there exists a solution of p and q that satisfies the Equation Set (2), then there exists a solution '$p = p^*, q = q^*$' that satisfies not only the Equation Set (2) but also one of the following two inequalities:

$$\frac{l_2}{2m+1} - \frac{\Delta_Q}{2} < q^* \le p^* < \frac{l_2}{2m+1} + \frac{\Delta_P}{2} \tag{5}$$

$$\frac{l_2}{2m+1} - \frac{\Delta_P}{2} \le p^* < q^* \le \frac{l_2}{2m+1} + \frac{\Delta_Q}{2} \tag{6}$$

Theorem 11: Let t be a positive even integer. Let $m = \frac{t}{2}$. Define A as

$$\begin{aligned} A \quad = \quad \max\{ \quad & (\lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \rceil + 1)m^2 + (3 - \lceil \tfrac{2l_2+(m+1)(2m+1)(2m^2+1)}{2l_2-m(2m+1)(2m^2+1)} \rceil)m - 3, \\ & (\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil + 1)m^2 + (3 - \lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil)m - 1 \\ & -2\lceil \tfrac{2l_2+m(2m+1)(2m^2+1)}{2l_2-(m+1)(2m+1)(2m^2+1)} \rceil \} \end{aligned}$$

. Then when

$$l_2 > \frac{(m+1)(2m+1)(2m^2+1)}{2}$$

and

$$l_1 \ge 2m^2 \left(\lceil \frac{A}{2m^2+1} \rceil (2m^2+1) - 2 \right)$$

, an $l_1 \times l_2$ (or equivalently, $l_2 \times l_1$) torus' t-interleaving number is either $|S_t|$ or $|S_t| + 1$.

We skip the specific construction of optimally t-interleaving large tori here, because of its similarity to Construction 4.3. But we present its sketch. Basically, if the torus can be perfectly t-interleaved, then it can be optimally t-interleaved using

Construction 2.2; if the torus cannot be perfectly t-interleaved and $t \geq 4$, then it can be optimally t-interleaved using the tori-tiling method. The only remaining case is 'the torus cannot be perfectly t-interleaved and $t = 2$'. In that case, we can optimally t-interleave the torus (say it is an $l_1 \times l_2$ torus) using $|S_t| + 1 = 3$ distinct integers in the following way: interleave a ring of l_1 vertices and a ring of l_2 vertices using 3 integers — 0, 1 and 2 — such that no two adjacent vertices in those two rings are assigned the same integer; for $i = 1, 2, \cdots, l_1$ (respectively, for $i = 1, 2, \cdots, l_2$), use $I(i)$ (respectively, use $J(i)$) to denote the integer assigned to the i-th vertex in the ring of l_1 (respectively, l_2) vertices; for $i = 0, 1, \cdots, l_1 - 1$ and $j = 0, 1, \cdots, l_2 - 1$, label the vertex (i, j) in the $l_1 \times l_2$ torus with the integer $(I(i+1) + J(j+1)) \bmod 3$ — and then the torus is optimally 2-interleaved.

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