

Homework # 2

Due Wednesday, February 1, 2006, at 1:30 PM

Collaboration is allowed and encouraged

This homework set focuses on Stone's 1936 celebrated Representation Theorem and Shannon's 1938 Masters Thesis. You are going to prove Stone's Theorem by solving problems 1 and 2. Problems 3 and 4 are based on Shannon's Masters Thesis, "A Symbolic Analysis of Relay and Switching Circuits". The thesis is posted on the class web site and we kindly suggest that you read this classical paper.

1. Boolean Subalgebras

Show that if $(A, 0_A, 1_A, +_A, \times_A)$ is a Boolean algebra, and $(B, 0_B, 1_B, +_B, \times_B)$ is a Boolean algebra, then $(C, 0_C, 1_C, +_C, \times_C)$ is also a Boolean algebra, where:

- $C = \{(a, b) \mid a \in A, b \in B\}$
- $0_C = (0_A, 0_B)$
- $1_C = (1_A, 1_B)$
- $+_C$ is defined by $(a_1, b_1) +_C (a_2, b_2) = (a_1 +_A a_2, b_1 +_B b_2)$
- $\times_C$ is defined by $(a_1, b_1) \times_C (a_2, b_2) = (a_1 \times_A a_2, b_1 \times_B b_2)$

(Note that it is ok if A is exactly the same as B .)

If $|A| = n$ and $|B| = m$, then what is $|C|$?

We will call the Boolean algebra on C the *product algebra* of the first two algebras.

Note: $|A|$ denotes the size of A , i.e., the number of elements in the set A . The operations $+_A$ and $\times_A$ denote the two operations of a Boolean algebra A .

2. Stone's Representation Theorem

In this problem we will show that all finite Boolean algebras have a certain form. Suppose we are given a Boolean algebra on B , with $|B| > 2$. We would like to express it as a product algebra of two smaller Boolean algebras. Pick an element of B , that is neither 0 nor 1. Call it s .

- (a) Show that there exists a Boolean algebra on the subset of B consisting of just those elements of B that are expressible as $b \cdot s$ for some $b \in B$.

(Note that you must specify a 0 element and a 1 element in addition to specifying the operations.)

- (b) Show that there exists a Boolean algebra on the subset of B consisting of just those elements of B that are expressible as $b \cdot \bar{s}$ for some $b \in B$. Why must the size of this subset be less than $|B|$? Why must the size be less than $|B| - 1$? Why must it be greater than 1?
- (c) We know from problem 1 that the product algebra of the two algebras found in 2a and 2b is a Boolean algebra. Show that the elements of this product algebra can be put in one-to-one correspondence with the elements of the original Boolean algebra B , so that even the $+$ and $\cdot$ operations correspond. (In this case we say that the two algebras are *isomorphic*, and the correspondence between them is called an *isomorphism*.)
- Note: "one-to-one correspondence" means that the elements of the two sets can all be paired up so that no two elements in either set are paired with the same element in the other set.*
- (d) In parts 2a-2c you showed that any Boolean algebra of 3 or more elements is isomorphic to a product algebra of smaller algebras. Does this tell us anything about what sizes (numbers of elements) are possible or impossible for B ?
- (e) How can you proceed from here to prove Stone's theorem?

Theorem 1 (Stone) *Any finite Boolean algebra is isomorphic to the bitvector Boolean algebra for some k (where k is the length of the bitvectors).*

Definition 1 *The bitvector Boolean algebra $(B, 0_B, 1_B, +_B, \times_B)$ for a positive integer k consists of*

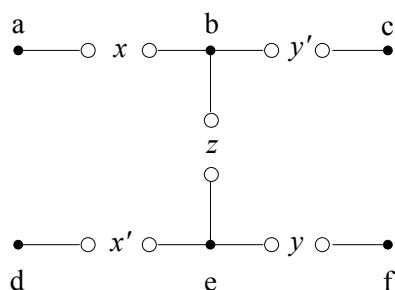
- $B = \{(b_1, b_2, \dots, b_k) \mid b_i \in \{0, 1\} \text{ for } 1 \leq i \leq k\}$
- $0_B = (0, 0, \dots, 0)$
- $1_B = (1, 1, \dots, 1)$
- $(x_1, x_2, \dots, x_k) +_B (y_1, y_2, \dots, y_k) = (x_1 \vee y_1, x_2 \vee y_2, \dots, x_k \vee y_k)$
where $x_i \in \{0, 1\}$ and $y_i \in \{0, 1\}$ for all $1 \leq i \leq k$.
- $(x_1, x_2, \dots, x_k) \times_B (y_1, y_2, \dots, y_k) = (x_1 \wedge y_1, x_2 \wedge y_2, \dots, x_k \wedge y_k)$
where $x_i \in \{0, 1\}$ and $y_i \in \{0, 1\}$ for all $1 \leq i \leq k$.

Note: $x \vee y$ is notation for OR(x, y), and $x \wedge y$ is notation for AND(x, y).

A comment related to problems 3 and 4: We use Shannon's convention: 0 denotes a closed circuit, 1 denotes an open circuit, $+$ (addition) denotes a series connection and $\cdot$ (multiplication) denotes a parallel connection. In your solutions, feel free to use the modern convention: 1 denotes a closed circuit, 0 denotes an open circuit, $\cdot$ denotes a series connection and $+$ denotes a parallel connection. However, state clearly which convention you are using.

3. A Generalization to Multiple Terminals

A generalization of the switching circuit model is a circuit with multiple terminals. As in Shannon's model, the hindrance X_{ab} is 0 if there is a closed path between terminals a and b , and 1 otherwise. With multiple terminals, $a, b, c, d, \dots$, hindrance functions can be measured between any pair of terminals. For instance, for the circuit



we have

$$X_{af} = x + y + z$$

$$X_{bd} = x' + z$$

$$X_{cf} = 1$$

and so on.

- (a) Construct a circuit with 3 switching elements that implements the functions

$$f_1 = xy$$

$$f_2 = x'y$$

- (b) Construct a circuit with 4 switching elements that implements the functions

$$f_1 = xy + x'y'$$

$$f_2 = x'y + xy'$$

- (c) Construct a circuit with 6 switching elements that implements the functions

$$f_1 = x(y + z)$$

$$\begin{aligned}
f_2 &= y(x + z) \\
f_3 &= z(x + y) \\
f_4 &= x + yz \\
f_5 &= y + xz \\
f_6 &= z + xy
\end{aligned}$$

(a solution with fewer than 6, if it exists, gets bonus points).

- (d) Construct a circuit with as few switching elements as possible that implements all functions of two variables. A solution with 8 switching elements gets full credit (a solution with fewer than 8, if it exists, gets bonus points).

4. Some Size Bounds on Switching Circuits

In Shannon's Masters Thesis, he explains how to develop a series expansion of an arbitrary boolean function as either a sum of products or a product of sums. (For this problem, use the method shown on the right hand side of Shannon's Figure 4, not the method shown in the center of Figure 4.)

- (a) Assume that we can show that any function of n variables requires no more than g_n switching elements. Show that any function of $n + 1$ variables can be implemented in at most $2g_n + 2$ elements.
- (b) Starting with $g_0 = 0$, use your method from part (a) to show that any boolean function can be implemented in a switching circuit with at most $\sum_{i=1}^n 2^i = 2^{n+1} - 2$ elements. This just corresponds to the full expansion of f around each of its variables which is just the minterm or maxterm representation of f .
- (c) By expanding around all but one of the variables of a function, we get the partial minterms of the first $n - 1$ variables, each coupled to a function of one variable. Since only two switching elements are required to implement all possible functions of one variable, show that any boolean function can be implemented with $2 + \sum_{i=1}^{n-1} 2^i = 2^n$ elements.
- (d) For larger n , it is possible to find a better bound by expanding about all but two variables.
- Construct a circuit using 20 switching elements that computes all 16 functions of two variables. (For bonus points, construct such a circuit with only 18 switching elements.)
 - Use this circuit to show that any function of $n > 2$ variables can be constructed using no more than $2^{n-1} + 18$ switching elements.
- (e) Show that it is possible to construct a circuit using no more than $\sum_{i=1}^m 2 \times 2^{2^i}$ switching elements that computes all 2^{2^m} functions of m variables. Reduce the number of switching elements to $\sum_{i=1}^m 2 \times (2^{2^i} - 2^{2^{i-1}})$, and finally down to $\sum_{i=1}^m 2 \times (2^{2^i} - 2^{2^{i-1}}) \frac{2^{2^{i-1}} - 1}{2^{2^{i-1}}}$.

- (f) Use your result from part (e) to show that any function of $n > 2$ variables can be constructed using no more than $2^{n-m+1} + \sum_{i=1}^m 2 \times (2^{2^{i-1}} - 1)^2$ switching elements, for any choice of m . What is the smallest n for which we should expand about three variables?