

CNS 188a

Computation Theory and Neural Systems

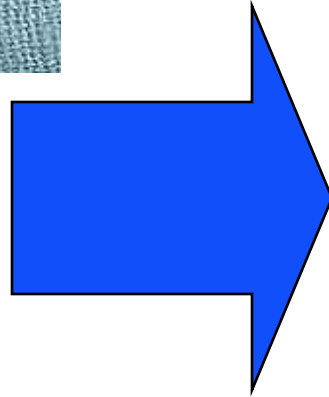
Monday and Wednesday 1:30-3:00
Moore 080

Lecturer: Shuki Bruck; 331 Moore
office hours: Mon, Wed, 3-4pm

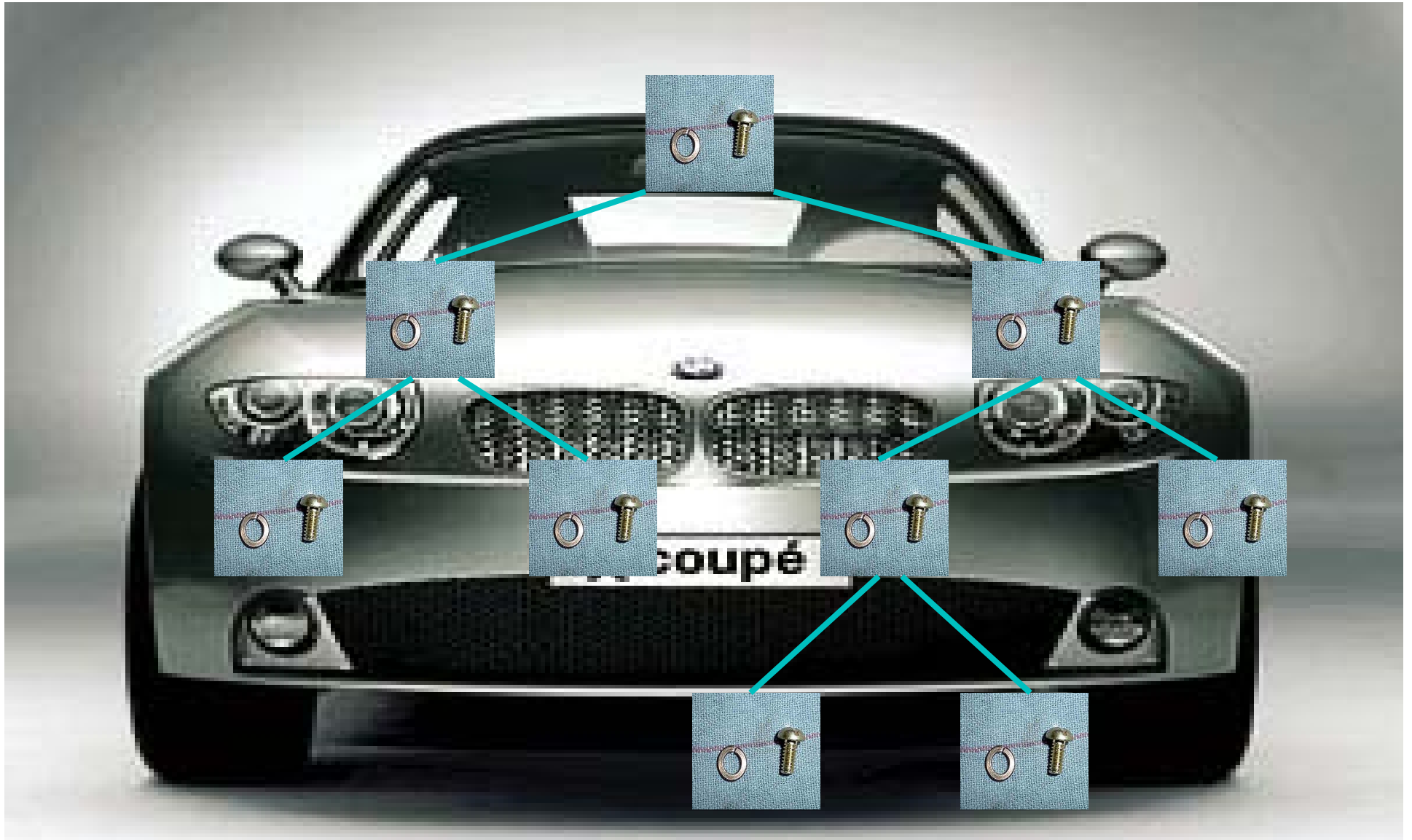
TAs: Vincent Bohossian, Matt Cook; 311 Moore
office hours: Mon, Tue, 8-9pm

Secretary: Michelle Chen; 304 Moore

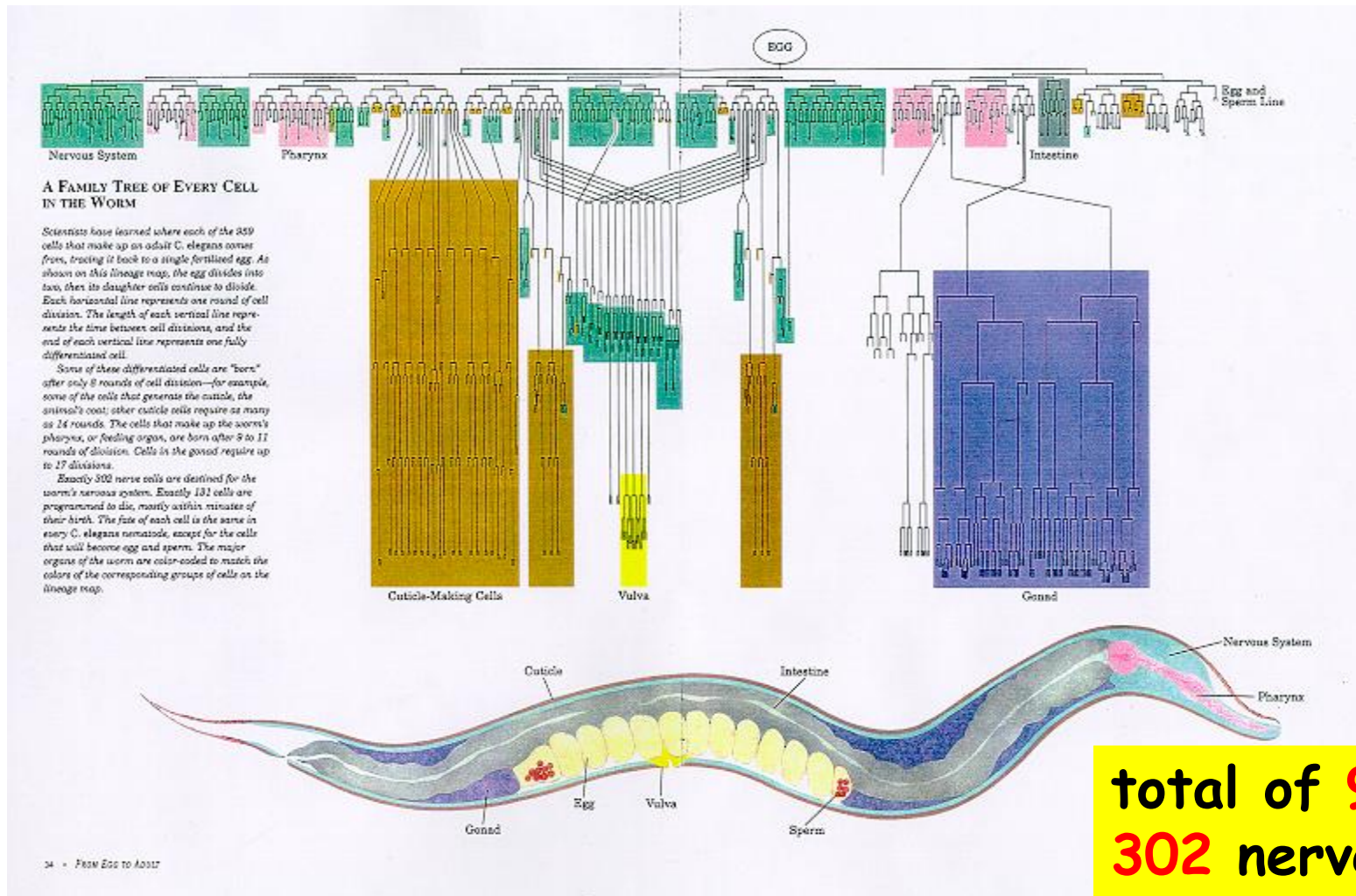
From Screws to Systems...



The Lineage of BMW



C. Elegans Lineage



total of **959** cells
302 nerve cells
131 cells are
destined to die

C. Elegans Lineage - Simple Questions

Dealing with **identity**:
How do cells remember what to do?



Dealing with **time**:
How do cells know when? No clock...

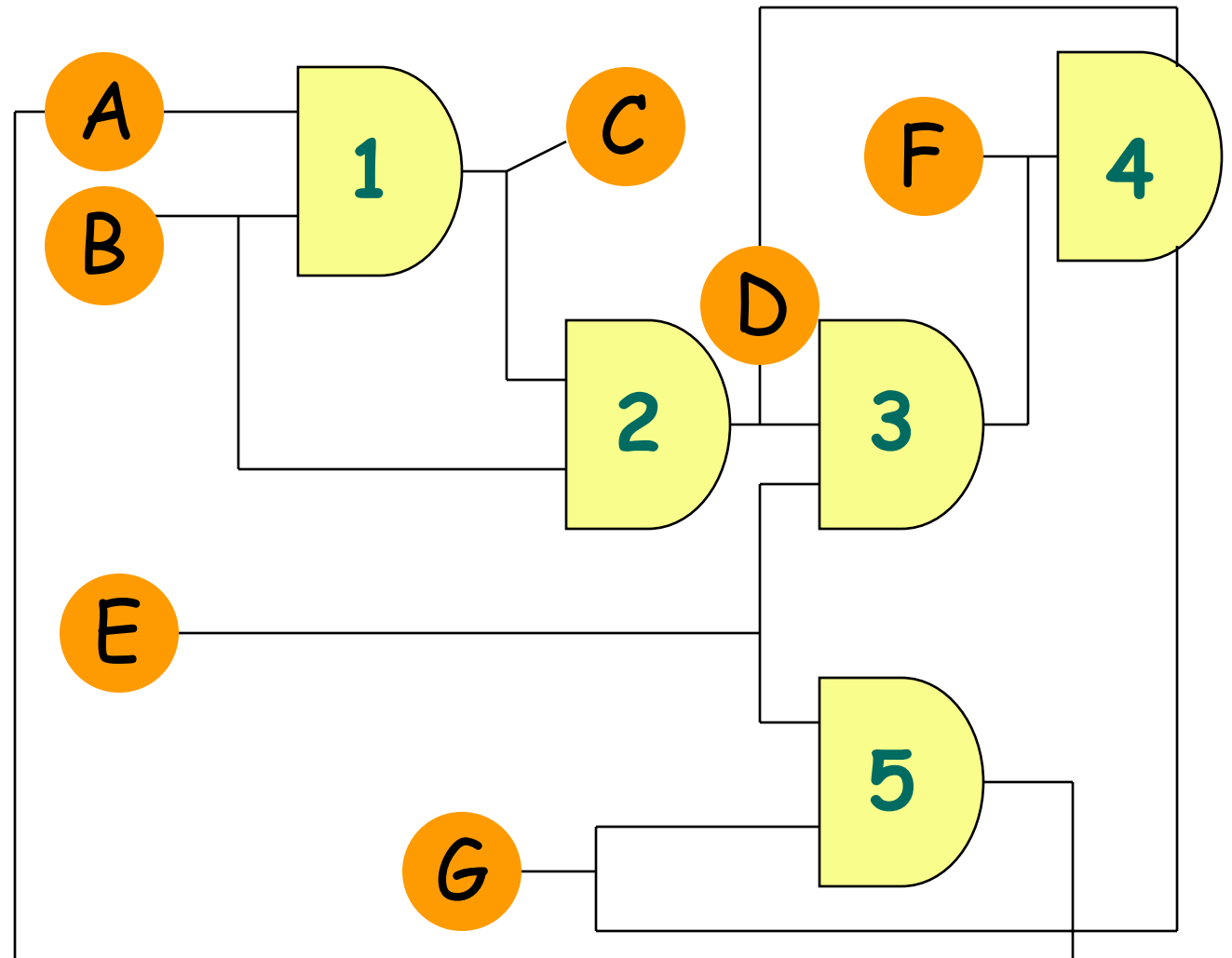
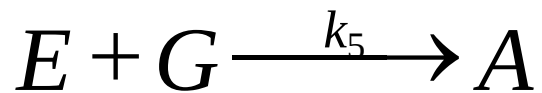
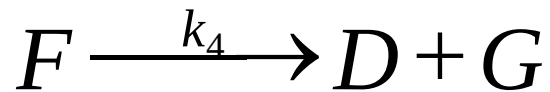
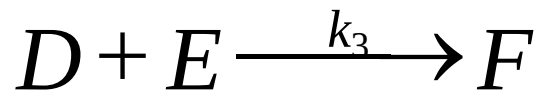
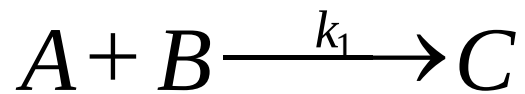


Dealing with **order**:
How do cells coordinate their actions?

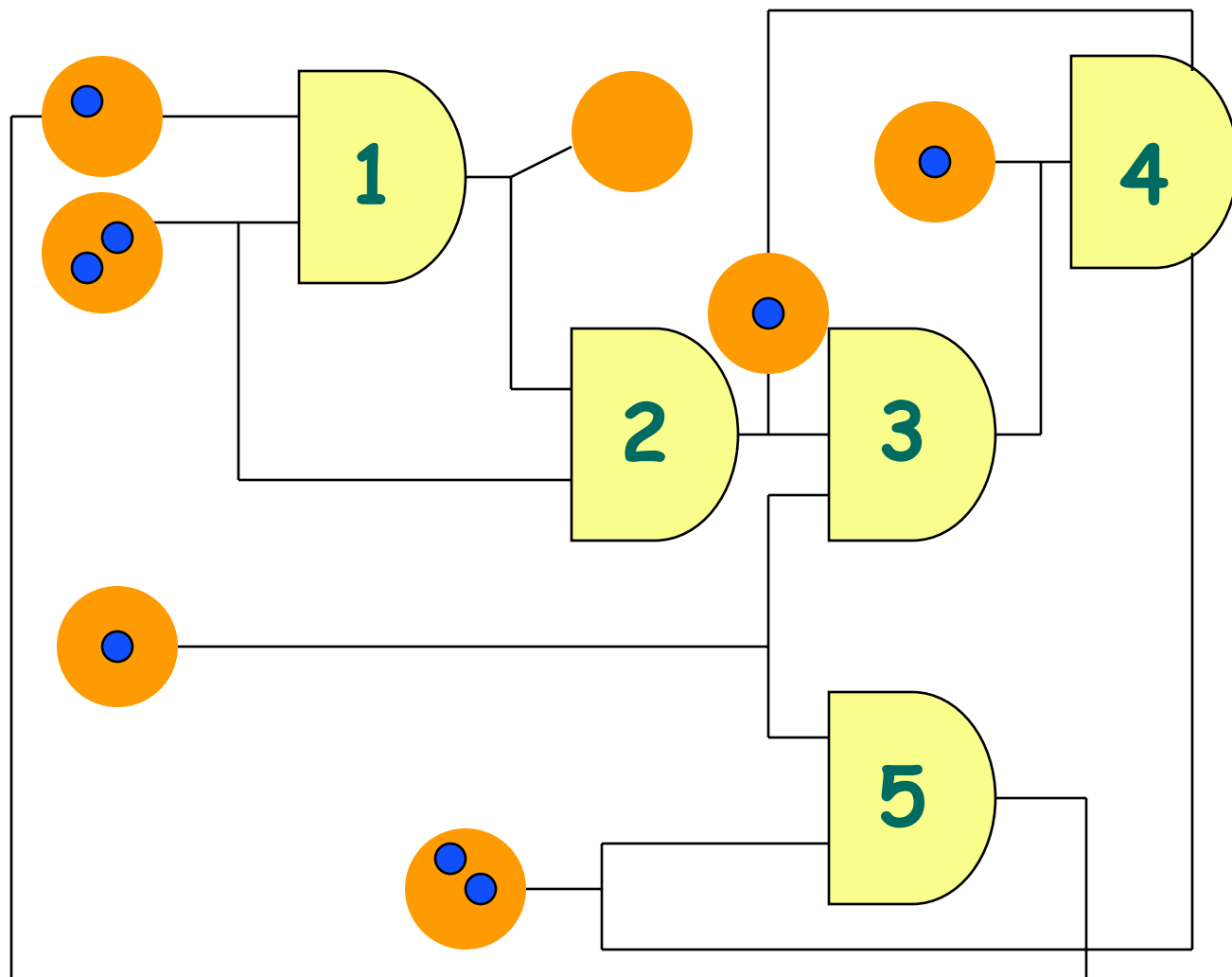
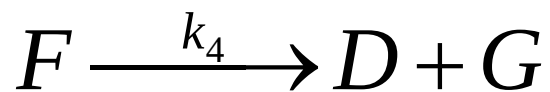


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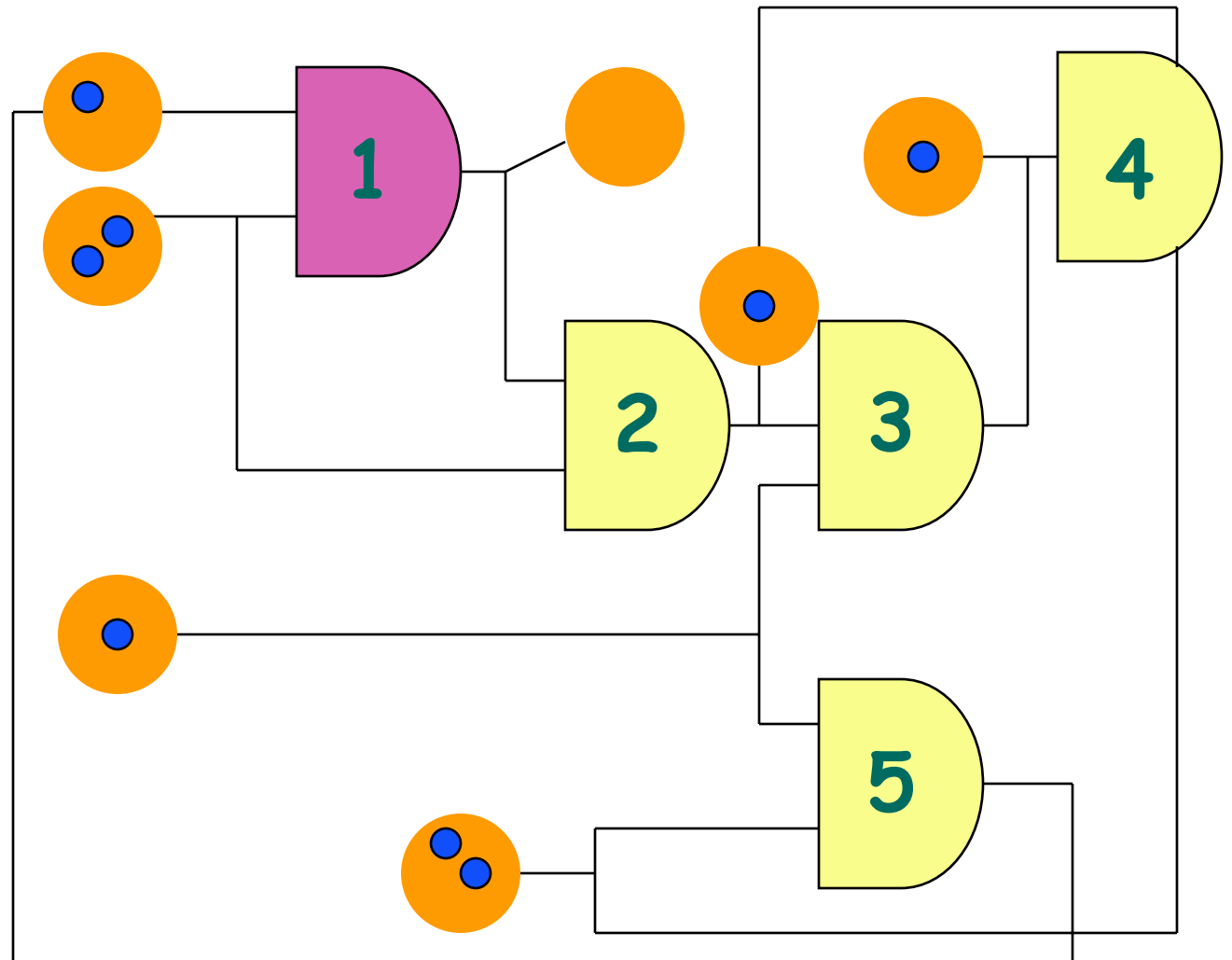
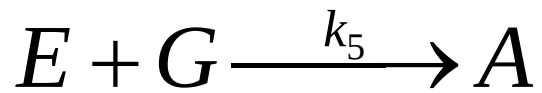
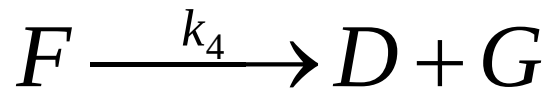
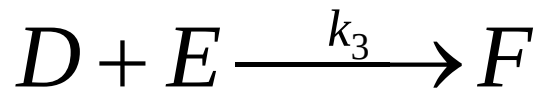
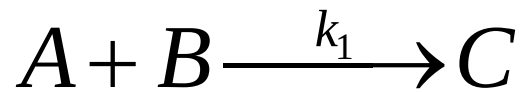
Control via Stochastic Chemical Reactions



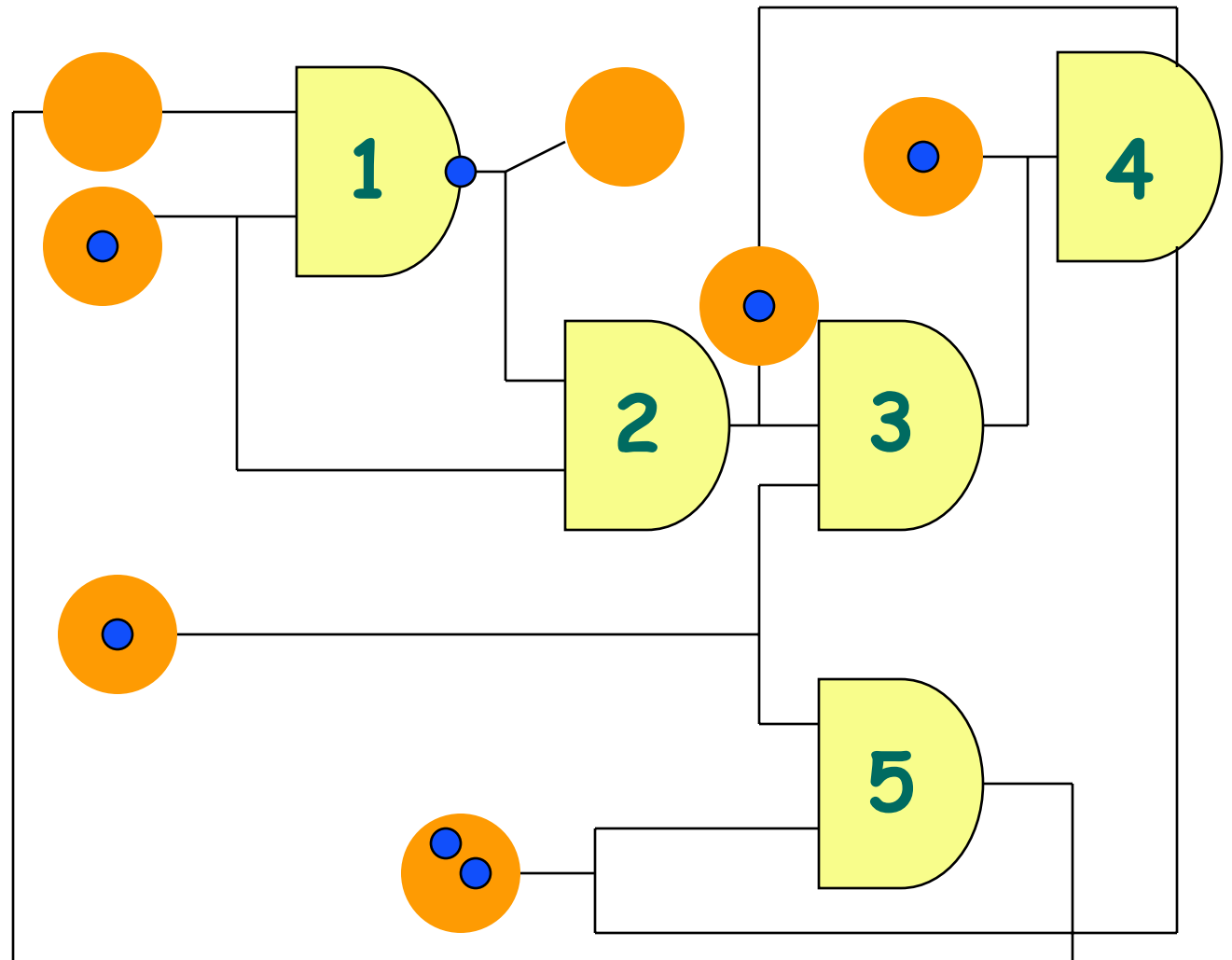
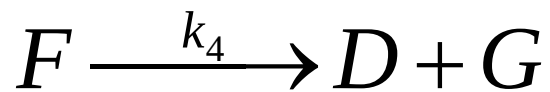
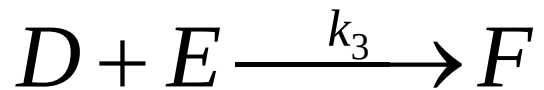
Chemical Reactions Circuits



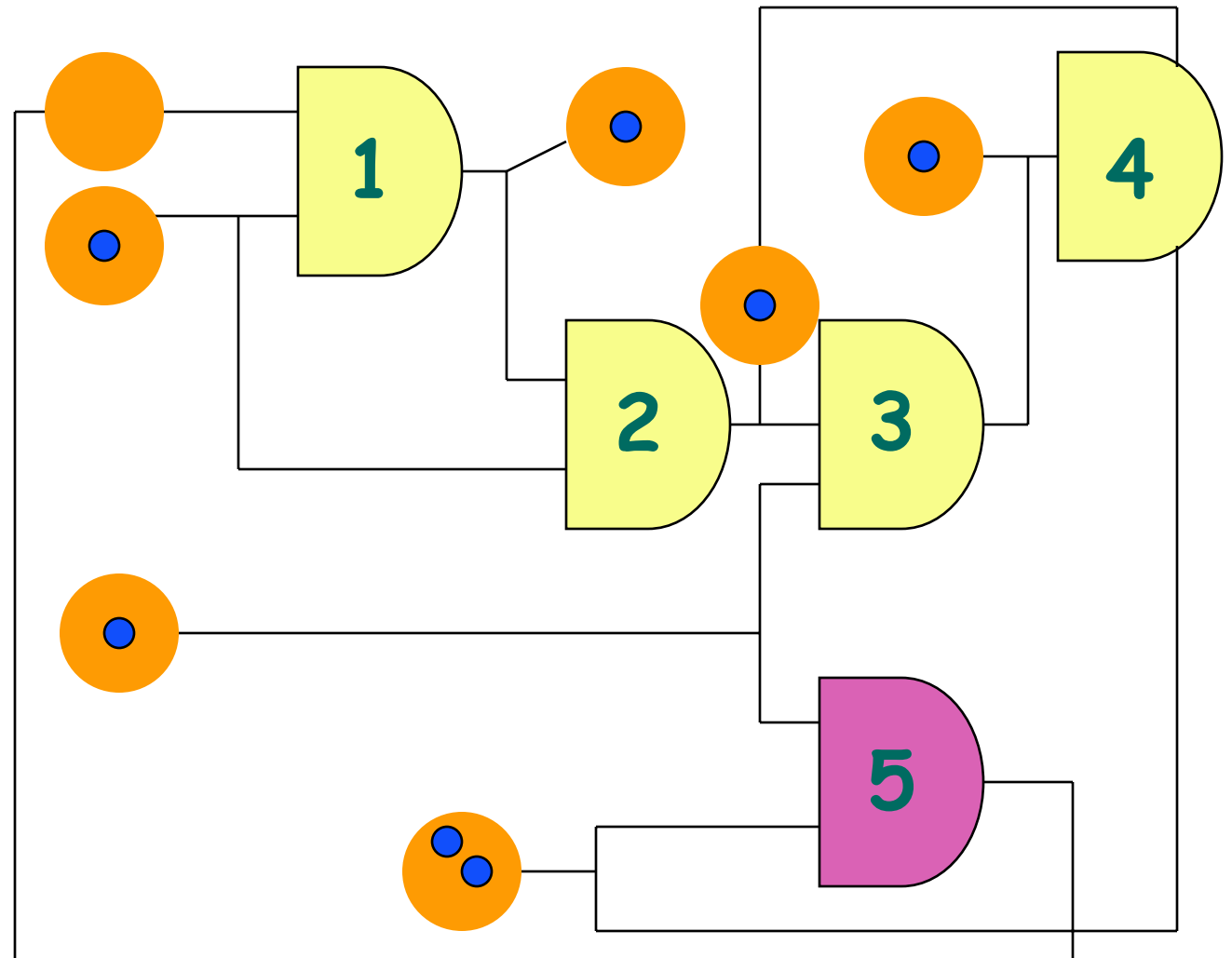
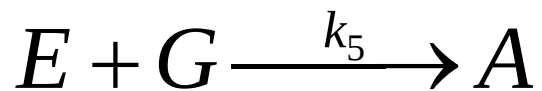
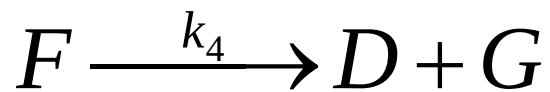
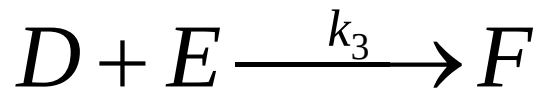
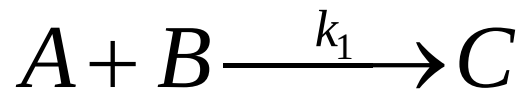
Chemical Reactions Circuits



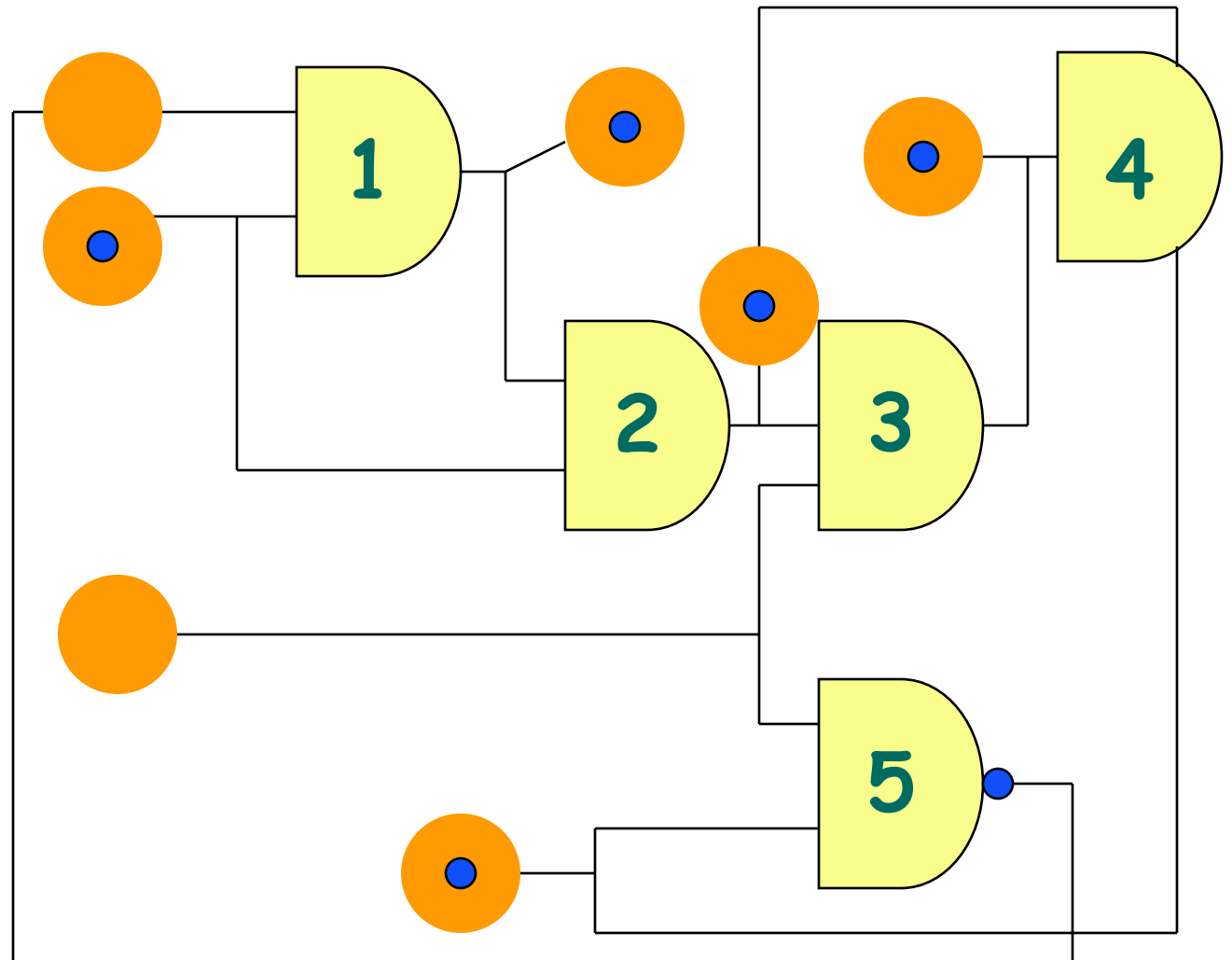
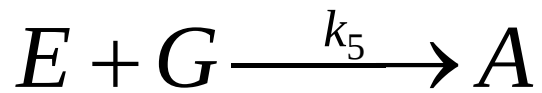
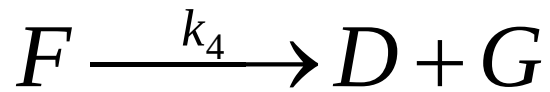
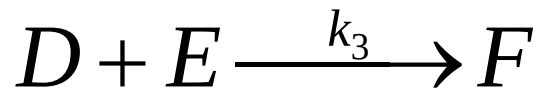
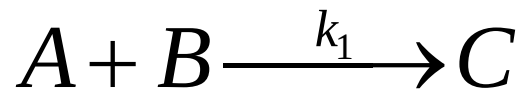
Chemical Reactions Circuits



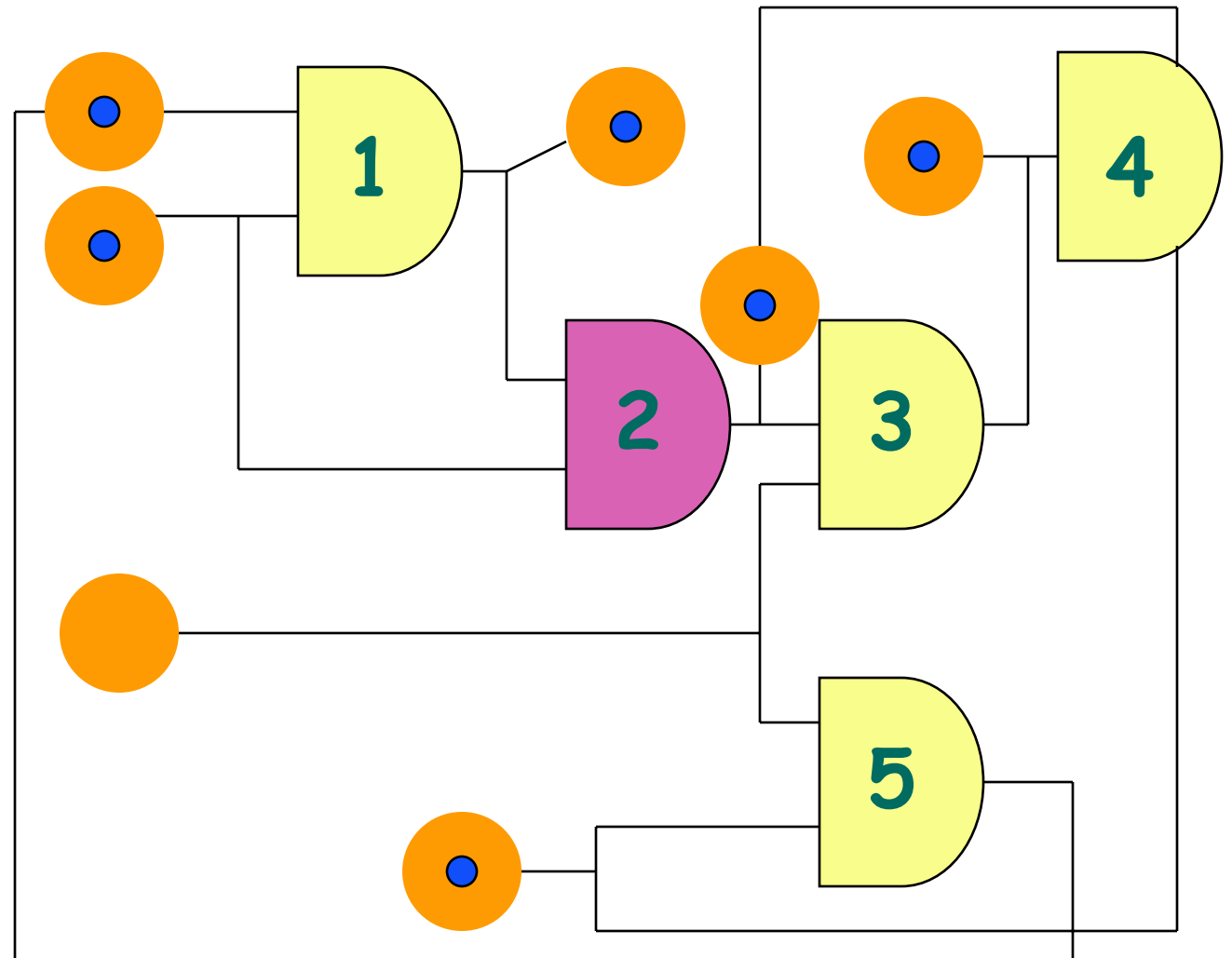
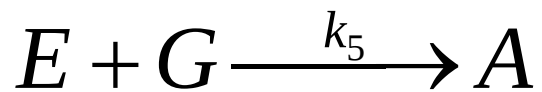
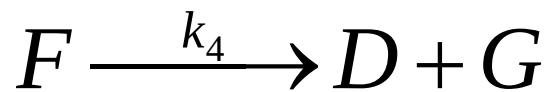
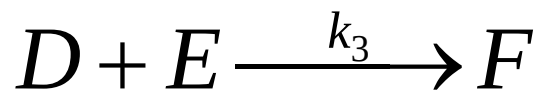
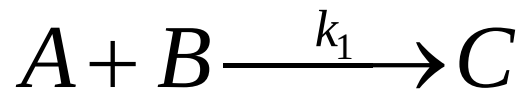
Chemical Reactions Circuits



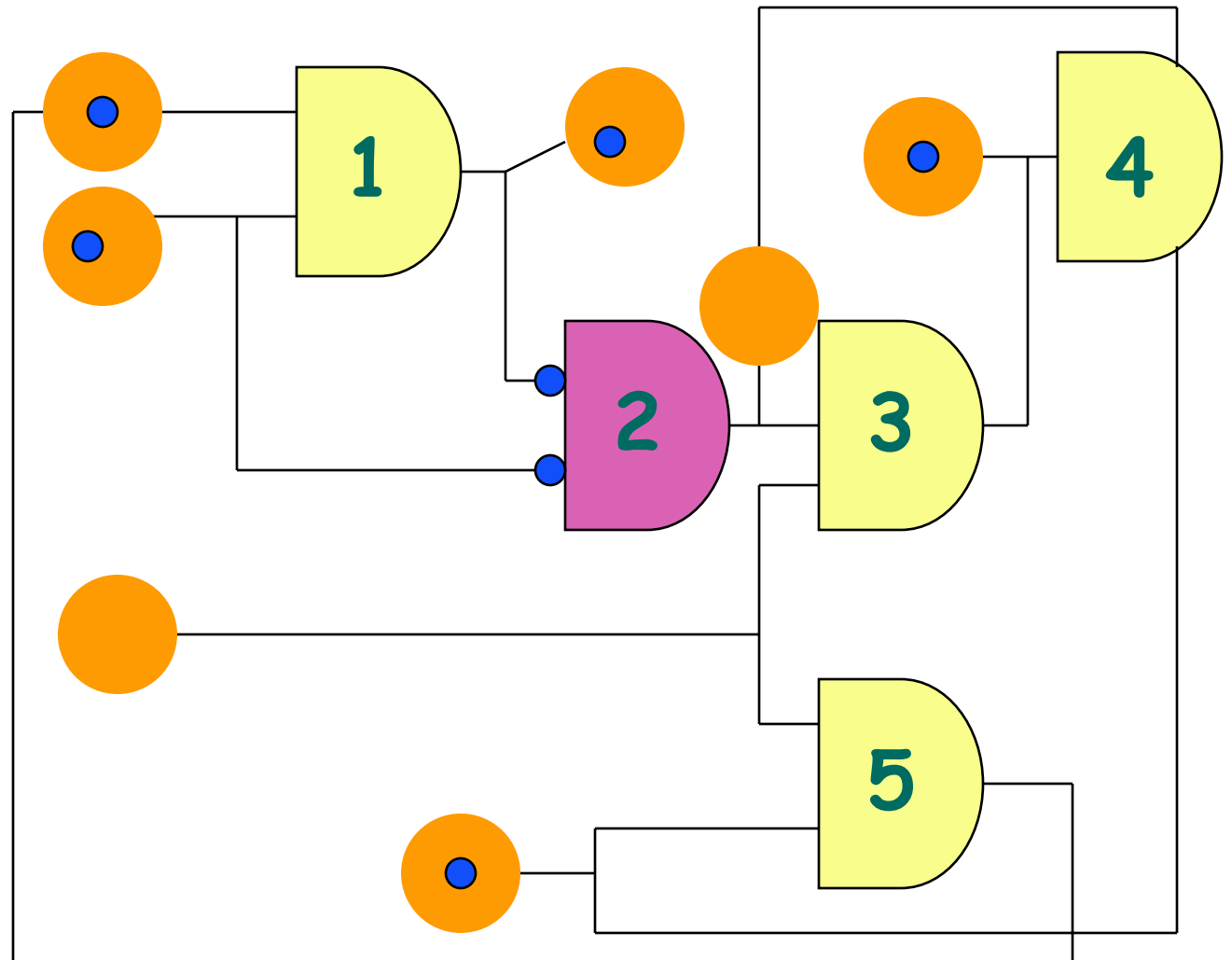
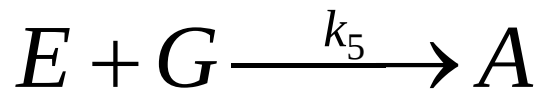
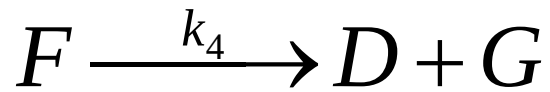
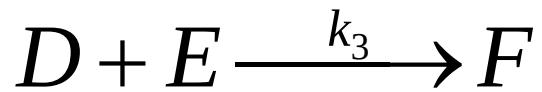
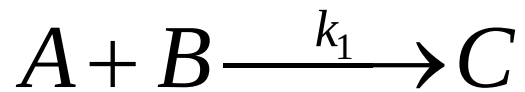
Chemical Reactions Circuits



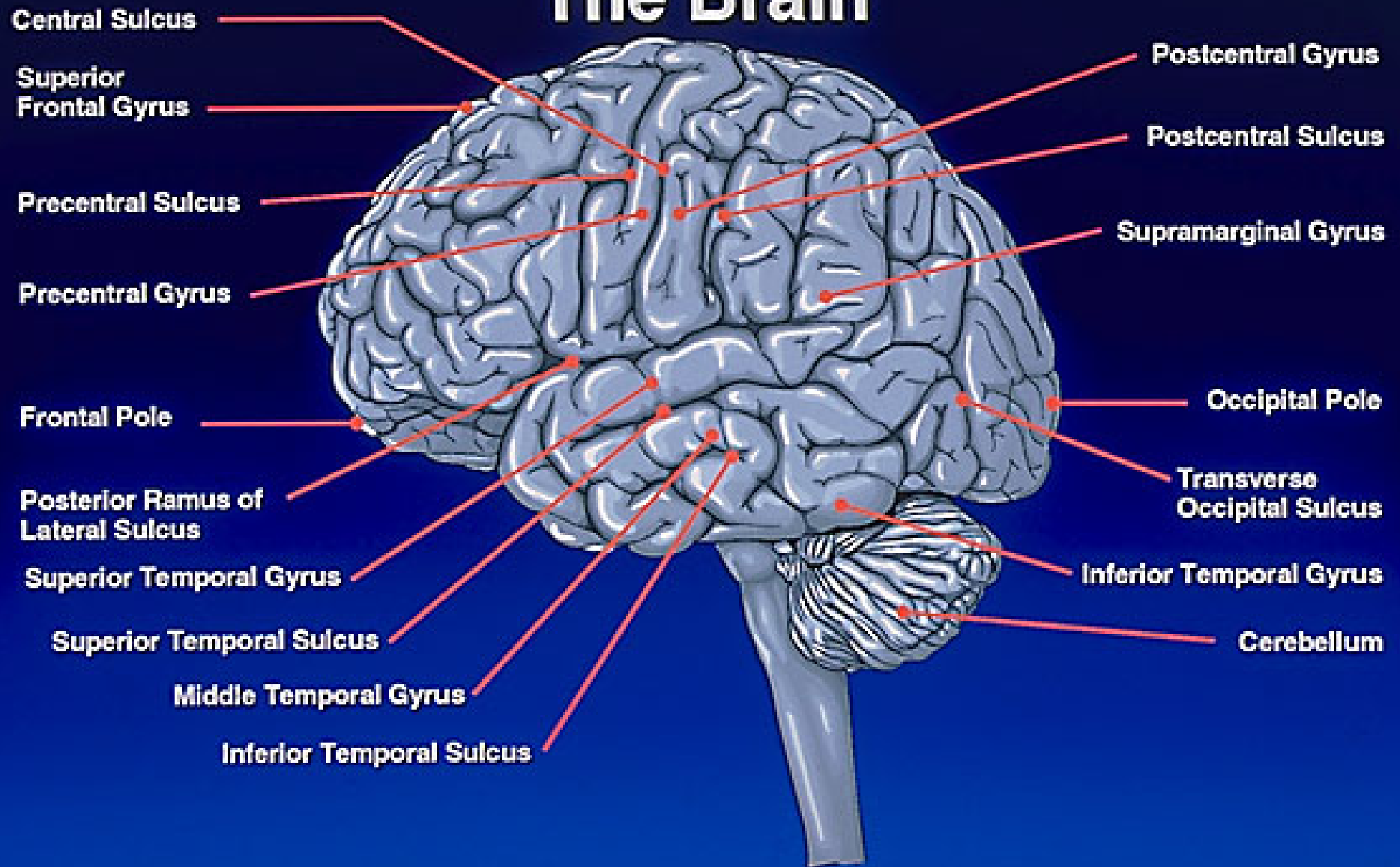
Chemical Reactions Circuits



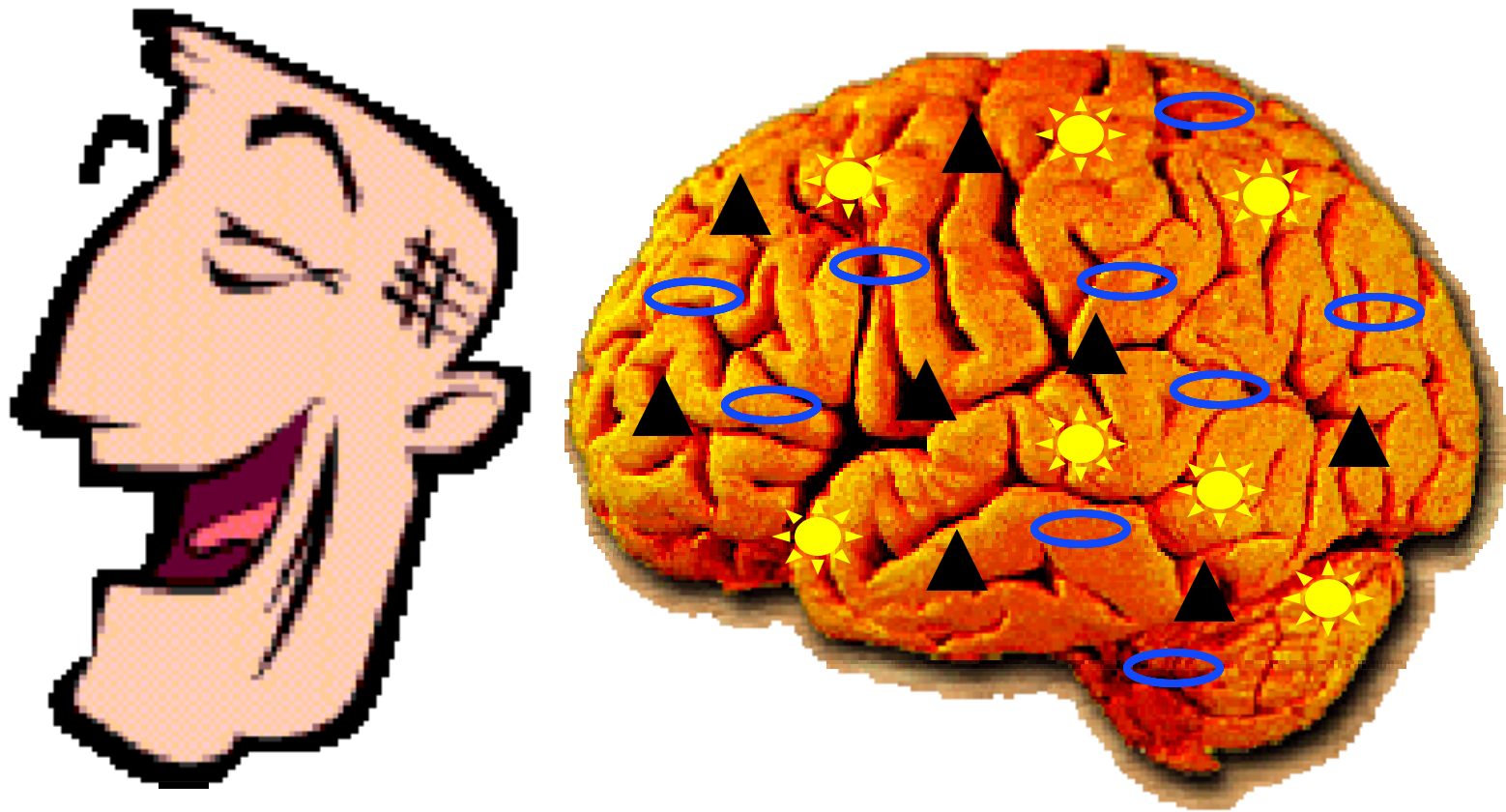
Chemical Reactions Circuits



The Brain



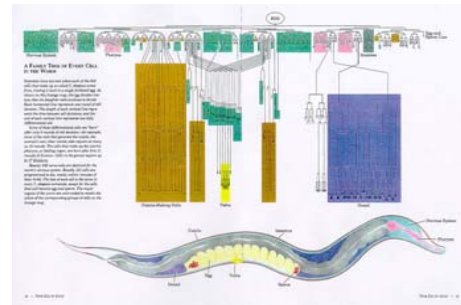
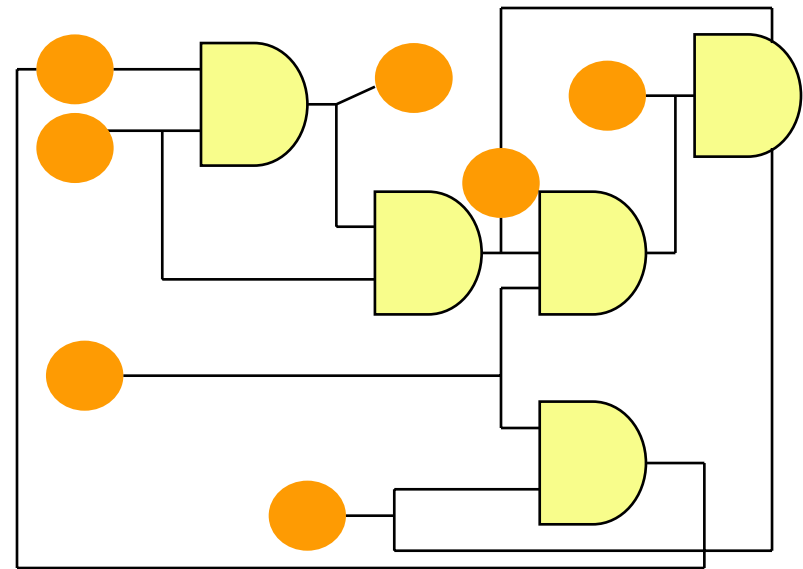
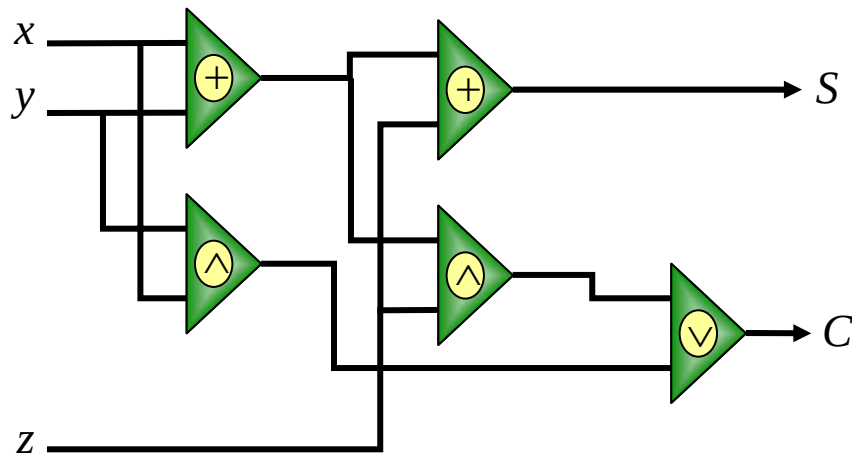
Descriptive Biology: Is It Enough?



A HUGE Gap between Ability to Design and Analyze

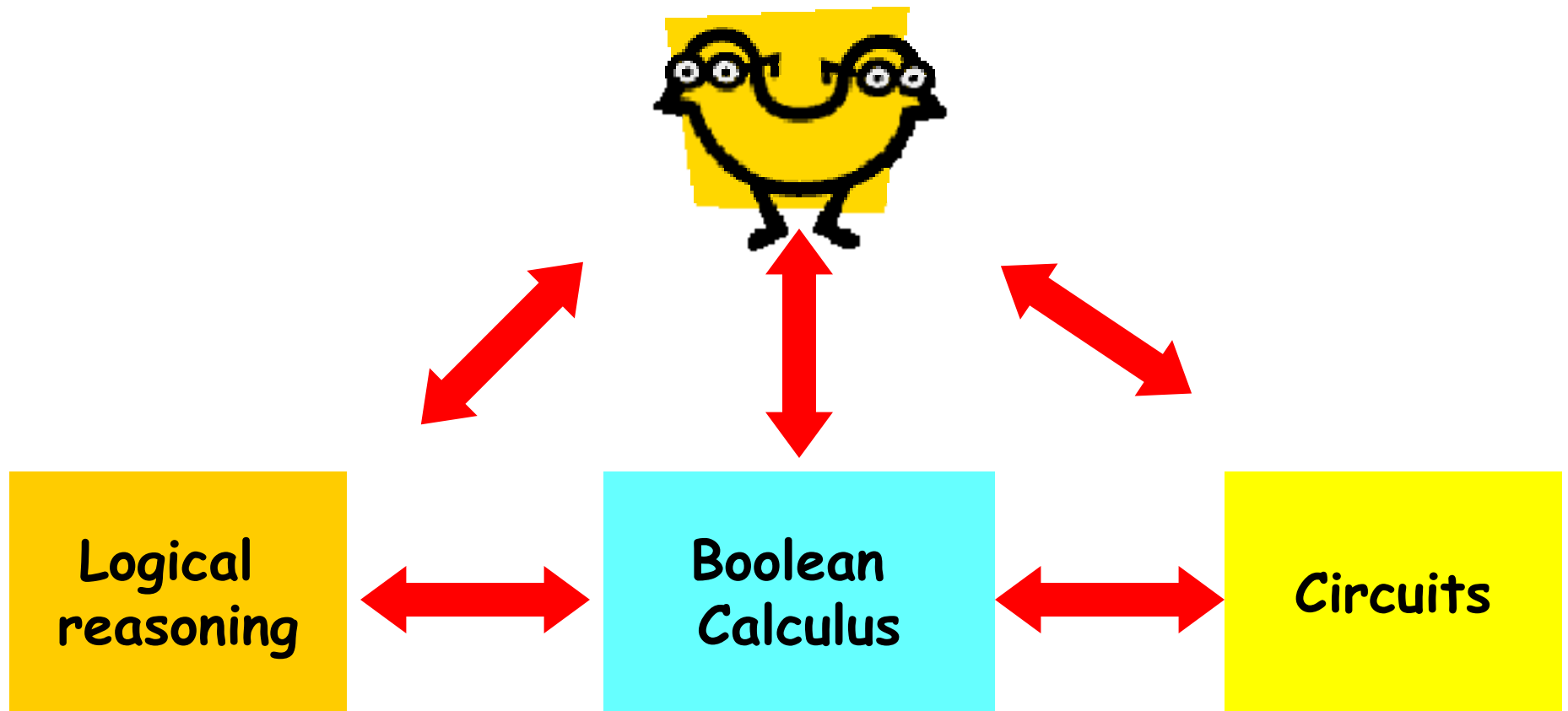
Design

Analysis



Key to the Progress in Design: Abstractions in Information Systems

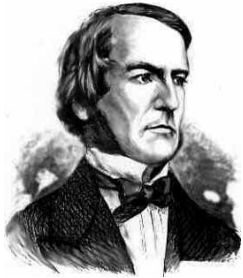
Reasoning **to** Calculations **to** Physical Circuits



Key to the Progress in Design: Abstractions in Information Systems

Logic to **Boolean Calculus** to Physical Circuits

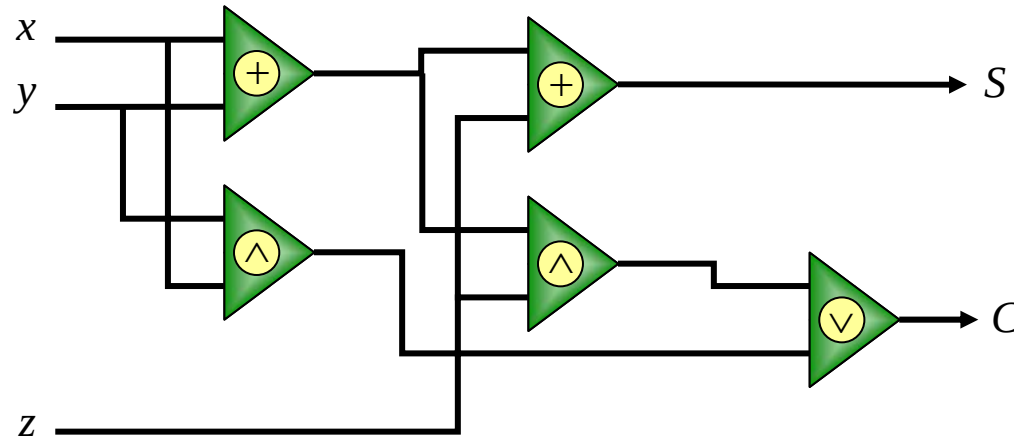
Boole
1815-1864



1847

Connected Logic
with Algebra

Boolean Algebra
Logical Calculation



Shannon
1916-2001



1938

Boolean Algebra to
Electrical Circuits
Logic Design

Text to Algebra

George Boole, 1854

GEORGE BOOLE

THE MATHEMATICAL ANALYSIS OF LOGIC

*With a new Introduction by
John Slater*

KEY TEXTS

Classic Studies in the History of Ideas

OF EXPRESSION AND INTERPRETATION.

25

the interpretation of v . But it has been thought better to write them separately, for greater ease and convenience. And it is further to be borne in mind, that although three different forms are given for the expression of each of the *particular* propositions, everything is really included in the first form.

TABLE.

The class X	x	
The class not-X	$1 - x$	
All Xs are Ys All Ys are Xs	$x = y$	
All Xs are Ys	$x(1 - y) = 0$	
No Xs are Ys	$xy = 0$	
All Ys are Xs Some Xs are Ys	$y = vx$	$vx = \text{some Xs}$ $v(1 - x) = 0.$
No Ys are Xs Some not-Xs are Ys	$y = v(1 - x)$	$v(1 - x) = \text{some not-Xs}$ $vx = 0.$
Some Xs are Ys	$\begin{cases} v = xy \\ \text{or } vx = vy \\ \text{or } vx(1 - y) = 0 \end{cases}$	$\begin{cases} v = \text{some Xs or some Ys} \\ vx = \text{some Xs, } vy = \text{some Ys} \\ v(1 - x) = 0, v(1 - y) = 0. \end{cases}$
Some Xs are not Ys	$\begin{cases} v = x(1 - y) \\ \text{or } vx = v(1 - y) \\ \text{or } vxy = 0 \end{cases}$	$\begin{cases} v = \text{some Xs, or some not-Ys} \\ vx = \text{some Xs, } v(1 - y) = \text{some not-Ys} \\ v(1 - x) = 0, vy = 0. \end{cases}$

Text to Algebra

George Boole, 1854

The class X

$$x$$

The class not-X

$$1 - x$$

All Xs are Ys }
All Ys are Xs }

$$x = y$$

All Xs are Ys

$$x(1 - y) = 0$$

No Xs are Ys

$$xy = 0$$

All Ys are Xs }
Some Xs are Ys }

$$y = vx$$

vx = some Xs

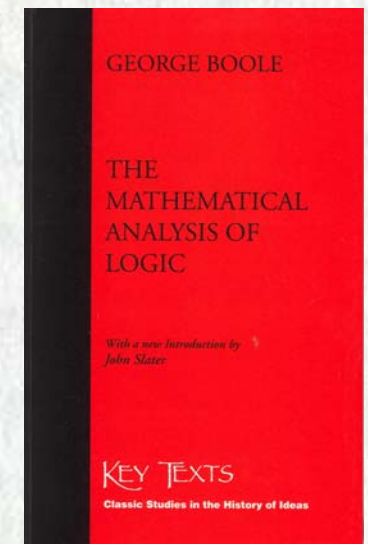
$$v(1 - x) = 0.$$

No Ys are Xs }
Some not-Xs are Ys }

$$y = v(1 - x)$$

$v(1 - x)$ = some not-Xs

$$vx = 0.$$



Shannon MSc Thesis, 1938

Algebra to Circuits

c_{k+1}	c_k	$c_{j+1}c_j$	c_2c_1	Carried numbers
	a_k	$\dots a_{j+1}a_j$	$\dots a_2a_1a_0$	First number
	b_k	$b_{j+1}b_j$	$b_2b_1b_0$	Second number
c_{k+1}	s_k	$\dots s_{j+1}s_j$	$\dots s_2s_1s_0$	Sum
or				
s_{k+1}				

Starting from the right, s_0 is one if a_0 is one and b_0 is zero or if a_0 is zero and b_0 one but not otherwise. Hence

$$s_0 = a_0 b'_0 + a'_0 b_0 = a_0 \oplus b_0 .$$

c_1 is one if both a_0 and b_0 are one but not otherwise:

$$c_1 = a_0 \cdot b_0 .$$

s_j is one if just one of a_j , b_j , c_j is one, or if all three are one:

$$s_j = S_{1,3}(a_j, b_j, c_j) , \quad j = 1, 2, \dots, k .$$

c_{j+1} is one if two or if three of these variables are one:

$$c_{j+1} = S_{2,3}(a_j, b_j, c_j) , \quad j = 1, 2, \dots, k .$$

Using the method of symmetric functions, and shifting down for s_j gives the circuits of Figure 35. Eliminating superfluous elements we arrive at Figure 36.

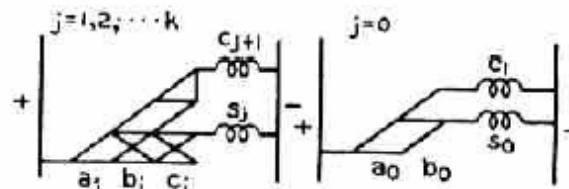


Figure 35. Circuits for electric adder

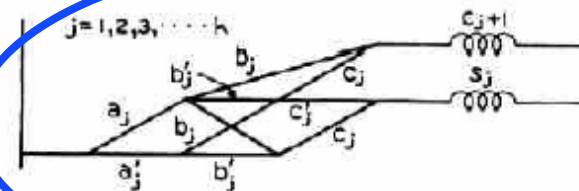


Figure 36. Simplification of figure 35

The Algebra (Boolean Calculus)

Boole, Jevons, Peirce, Schroder (18xx)
Axiomatic System: Huntington (1904)

Algebraic system: set of elements B ,
two binary operations $+$ and $\cdot$
 B has at least two elements (0 and 1)

If the following axioms are true
then it is a **Boolean Algebra**:

The Algebra (Boolean Calculus)

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Algebraic system: set of elements B ,
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 B has at least two elements (0 and 1)

If the following axioms are true
then it is a **Boolean Algebra**:

A1. identity

$$a + 0 = a; \quad a \cdot 1 = a$$

A2. complement

$$a + \bar{a} = 1; \quad a \cdot \bar{a} = 0$$

A3. commutative

$$a + b = b + a; \quad a \cdot b = b \cdot a$$

A4. distributive

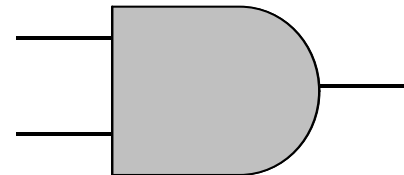
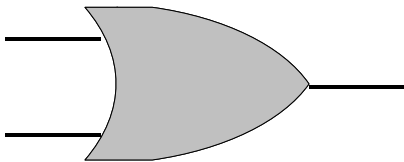
$$a + b \cdot c = (a + b) \cdot (a + c); \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

xy	$OR(x,y)$
00	0
01	1
10	1
11	1

xy	$AND(x,y)$
00	0
01	0
10	0
11	1



Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

The following axioms are **obviously** true:

$$a + 0 = a; \quad a \cdot 1 = a$$

$$a + \bar{a} = 1; \quad a \cdot \bar{a} = 0$$

$$a + b = b + a; \quad a \cdot b = b \cdot a$$

$$a + b \cdot c = (a + b) \cdot (a + c); \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

A1. identity

$$a + 0 = a; \quad a \cdot 1 = a$$

$$\begin{array}{l} 0 + 0 = 0 \\ 1 + 0 = 1 \end{array}$$

$$\begin{array}{l} 0 \cdot 1 = 0 \\ 1 \cdot 1 = 1 \end{array}$$

xy	OR(x,y)
00	0
01	1
10	1
11	1

xy	AND(x,y)
00	0
01	0
10	0
11	1

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

A2. complement

$$a + \bar{a} = 1; \quad a \cdot \bar{a} = 0$$

$$0 + 1 = 1$$

$$1 + 0 = 1$$

$$0 \cdot 1 = 0$$

$$1 \cdot 0 = 0$$

xy	OR(x,y)
00	0
01	1
10	1
11	1

xy	AND(x,y)
00	0
01	0
10	0
11	1

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

A3. commutative $a + b = b + a; \quad a \cdot b = b \cdot a$

$$0+0 = 0+0$$

$$0+1 = 1+0$$

$$1+0 = 0+1$$

$$1+1 = 1+1$$

$$0 \cdot 0 = 0 \cdot 0$$

$$0 \cdot 1 = 1 \cdot 0$$

$$1 \cdot 0 = 0 \cdot 1$$

$$1 \cdot 1 = 1 \cdot 1$$

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
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A4. distributive

$$a + b \cdot c = (a + b) \cdot (a + c); \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

A4. distributive

$$a + b \cdot c = (a + b) \cdot (a + c); \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

$$0+0 \cdot 0 = (0+0) \cdot (0+0) = 0$$

$$0+0 \cdot 1 = (0+0) \cdot (0+1) = 0$$

$$0+1 \cdot 0 = (0+1) \cdot (0+0) = 0$$

$$0+1 \cdot 1 = (0+1) \cdot (0+1) = 1$$

$$1+0 \cdot 0 = (1+0) \cdot (1+0) = 1$$

$$1+0 \cdot 1 = (1+0) \cdot (1+1) = 1$$

$$1+1 \cdot 0 = (1+1) \cdot (1+0) = 1$$

$$1+1 \cdot 1 = (1+1) \cdot (1+1) = 1$$

Historical notes: Boolean Algebra Defined by Axioms (Postulates)

Pre-Boole (16xx): Leibniz; universal language for reasoning?

Beginning (18xx): Boole, Jevons, Peirce, Schroder, Whitehead

Next step: Huntington 1904; improved set of axioms

Sheffer 1913: Five axioms and one binary operation

Huntington/Robbins 1933 - three axioms, conjecture

Recent progress : McCune 1996; Robbins conjecture proved!

Historical notes: Three Axioms

- **A1. Commutativity:** $a + b = b + a$
- **A2. Associativity:** $(a + b) + c = a + (b + c)$
- **A3a. Huntington:** $a = \overline{(\bar{a} + b)} + \overline{(\bar{a} + \bar{b})}$
- **A3b. Robbins:** $a = \overline{\overline{(a + b)} + \overline{(a + \bar{b})}}$

Back to the Axioms

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

Q1: is the complement unique / well defined?

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

One Way to Say No!

- **A1. Identities:**
 $a + 0 = a$ and $a \cdot 1 = a$
- **A2. Complements:**
 $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- **A3. Commutativity:**
 $a + b = b + a$ and $a \cdot b = b \cdot a$
- **A4. Distributivity:**
 $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

Theorem 1:

Each element of a Boolean Algebra has exactly one complement.

Proof:

First we will prove that an element is **not self-complement**

Self Absorption

- A1. Identities:
 $a + 0 = a$ and $a \cdot 1 = a$
- A2. Complements:
 $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- A3. Commutativity:
 $a + b = b + a$ and $a \cdot b = b \cdot a$
- A4. Distributivity:
 $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

Lemma 1:

$$a + a = a$$

$$a \cdot a = a$$

Proof:

$$a + a = (a + a) \cdot 1$$

$$= (a + a)(a + \bar{a})$$

$$= a + a \cdot \bar{a}$$

$$= a + 0$$

$$= a$$

A1

A2

A4

A2

A1

Q



SpinOut – Keister



William Keister (1907-1997) was a pioneer in switching theory and design at Bell Labs. When he retired in 1972, he was director of Bell Labs' Computing Technology Center at Holmdel, New Jersey.

Mr. Keister began working in his spare time to prove that puzzles could be solved using Boolean algebra.



U.S. Patent 3637215 (1972): SpinOut

U.S. Patent 3637216 (1972): The Hexadecimal Puzzle

One Way to Say No!

- **A1. Identities:**
 $a + 0 = a$ and $a \cdot 1 = a$
- **A2. Complements:**
 $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- **A3. Commutativity:**
 $a + b = b + a$ and $a \cdot b = b \cdot a$
- **A4. Distributivity:**
 $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

Theorem 1:

Each element of a Boolean Algebra has exactly one complement.

Proof:

First we will prove that an element is **not self-complement**

Assume that: $a = \bar{a}$

By Lemma 1: $a + a = a$

$$a + \bar{a} = a$$

However by A2: $a + \bar{a} = 1$

 $a = 1$

Contradiction!

By A1: $1 \cdot 1 = 1$

By A2: $a \cdot \bar{a} = 0$

Q

One Way to Say No!

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Theorem 1:

Each element of a Boolean Algebra has exactly one complement.

Proof: We proved that an element is **not self-complement**

Next will prove that the complement is **unique**

One Way to Say No!

Proof:

- **A1. Identities:**
 $a + 0 = a$ and $a \cdot 1 = a$
- **A2. Complements:**
 $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- **A3. Commutativity:**
 $a + b = b + a$ and $a \cdot b = b \cdot a$
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Need to prove that the complement is **unique**

By contradiction: Assume an element has two **distinct** complements

$$a + \bar{a} = 1$$

$$a + x = 1$$

$$a \cdot \bar{a} = 0$$

$$a \cdot x = 0$$

x	$=$	$x \cdot 1$		$=$	$\bar{a} \cdot a + \bar{a} \cdot x$
	$=$	$x \cdot (a + \bar{a})$		$=$	$\bar{a} \cdot (a + x)$
	$=$	$x \cdot a + x \cdot \bar{a}$		$=$	$\bar{a} \cdot 1$
	$=$	$a \cdot x + \bar{a} \cdot x$		$=$	$\bar{a}$
	$=$	$0 + \bar{a} \cdot x$			

Contradiction!

Back to the Axioms

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

T1: one complement per element

$$\mathbf{L1:} \quad a + a = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

Is $a \cdot a = a$??

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

Yes: by Principle of Duality

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Duality

- A1. Identities:
 $a + 0 = a$ and $a \cdot 1 = a$
- A2. Complements:
 $a + \bar{a} = 1$ and $a \cdot \bar{a} = 0$
- A3. Commutativity:
 $a + b = b + a$ and $a \cdot b = b \cdot a$
- A4. Distributivity:
 $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

Theorem 0:

Any identity that is true in a Boolean algebra, is also true if $+$ and $\cdot$ are interchanged, and 0 and 1 are interchanged.

Proof:

It is true for the axioms!

Duality

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Lemma 1: $a + a = a$

$$a \cdot a = a$$

Proof:

$$\begin{aligned} a + a &= (a + a) \cdot 1 \\ &= (a + a)(a + \bar{a}) \\ &= a + a \cdot \bar{a} \\ &= a + 0 \\ &= a \end{aligned}$$

$$\begin{aligned} a \cdot a &= (a \cdot a) + 0 \\ &= (a \cdot a) + (a \cdot \bar{a}) \\ &= a \cdot (a + \bar{a}) \\ &= a \cdot 1 \\ &= a \end{aligned}$$

So far... True for any Boolean Algebra

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

T0: duality principle

T1: one complement per element

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

L1: $a + a = a$

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Non-Binary Boolean Algebras

Boolean Integers (Bunitskiy 1899)

$$2 \times 3 \times 5 = 30$$

$$2 \times 3 \times 7 = 42$$

Every prime in the prime factorization
is a power of one

The set of divisors of a Boolean integer

$\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

Non-Binary Boolean Algebras

The set $\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

$$6 + 15 = \text{lcm}(6, 15) = ?$$

$$6 \cdot 15 = \text{gcd}(6, 15) = ?$$

Non-Binary Boolean Algebras

The set $\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

$$6 + 15 = \text{lcm}(6, 15) = 30$$

$$6 \cdot 15 = \text{gcd}(6, 15) = 3$$

What is the Complement?

- A1. Identities:
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 $a + b \cdot c = (a + b) \cdot (a + c)$ and $a \cdot (b + c) = a \cdot b + a \cdot c$

The set $\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

$$a + \bar{a} = \text{lcm}(a, \bar{a}) = 30$$

$$a \cdot \bar{a} = \text{gcd}(a, \bar{a}) = 1$$

CNS 188a Overview

- **Boolean algebra as an axiomatic system**
- Boolean **functions** and their **representations** using Boolean formulas and spectral methods
- **Implementing** Boolean functions with circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the **complexity** (size and depth) of circuits
- **Relations** (as opposed to functions) and their implementation in circuits
- **Feedback** and convergence in LT circuits