

CNS 188a Overview

- Boolean algebra as an axiomatic system
- Boolean functions and their representations using Boolean formulas and spectral methods
- Implementing Boolean functions with relay circuits, circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits

The star of the Show: Polynomial Representation

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

$$f(1, 1) = \frac{1}{2}(1 + 1 + 1 - 1) = 1$$

$$f(1, -1) = \frac{1}{2}(1 + 1 - 1 + 1) = 1$$

$$f(-1, 1) = \frac{1}{2}(1 - 1 + 1 + 1) = 1$$

$$f(-1, -1) = \frac{1}{2}(1 - 1 - 1 - 1) = -1$$

AND

Polynomial Representation

Works for AND, OR and ...

AND

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

OR

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

NOT

$$f(x_1) = -x_1$$

XOR

$$f(x_1, x_2, \dots, x_n) = x_1x_2 \cdots x_n$$

Polynomial Representation

Works for any Boolean Function

Q: It works for AND, OR and NOT
Does it work for any Boolean function?
Is it unique?

AND

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

OR

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

NOT

$$f(x_1) = -x_1$$

Spectral Analysis of Boolean Functions

Notation

An efficient way to represent a multilinear monomial (term)

$$x^2 = 1 \text{ for } x \in \{-1, 1\}$$

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\alpha \in \{0, 1\}^n$$

For example:

$$\alpha = 11 \longrightarrow X^\alpha = x_1 x_2$$

$$\alpha = 00 \longrightarrow X^\alpha = 1$$

Spectral Analysis of Boolean Functions Notation

An efficient way to represent a multilinear monomial (term)

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \quad \alpha \in \{0, 1\}^n$$

The AND example:

$$f(x_1, x_2) = \frac{1}{2}(X^{00} + X^{10} + X^{01} - X^{11})$$

$\downarrow$

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

The coefficients:

$$a_{00} \quad a_{10} \quad a_{01} \quad a_{11}$$

Polynomial Representation Theorem

Representation Theorem:

(Fourier 18xx, Plancherel 19xx, Muller 195x, Ninomiya 195x)

(i) Every Boolean function has a unique representation as a multilinear polynomial of the form:

$$X = (x_1, x_2, \dots, x_n)$$

$$f : \{1, -1\}^n \mapsto \{1, -1\}$$

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

spectrum of the function

Polynomial Representation Theorem

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Sylvester type
Hadamard matrix

The coefficients listed in lexicographic order

The function listed in lexicographic order

Sylvester Type Hadamard Matrix

J. J. Sylvester, *Thoughts on Inverse Orthogonal Matrices, Simultaneous Sign Successions, and Tessellated Pavements in Two or More Colors*, Phil, Mag., 34, pp. 461–475, 1867.

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Recursive definition of H :

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

Polynomial Representation Theorem

Example

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} f_n$$

The AND function:

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

Polynomial Representation Theorem

Example

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} f_n$$

The OR function:

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

Polynomial Representation Theorem

Proof

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Proof: ??

Idea: we will compute the spectrum and show that the solution is unique

How to compute the spectrum?

Idea: solve a system of equations...

Polynomial Representation Theorem

Proof

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Proof: **Idea:** we will compute the spectrum by solving a system of equations

$n=1$

$$f(x_1) = a_0 + a_1 x_1 \begin{array}{l} \longrightarrow f(1) = a_0 + a_1 \\ \searrow f(-1) = a_0 - a_1 \end{array} \quad \text{linear}$$



$$a_0 = \frac{f(1) + f(-1)}{2}$$

$$a_1 = \frac{f(1) - f(-1)}{2}$$

Polynomial Representation Theorem

Proof

Representation Theorem:

(ii) The spectrum can be computed as follows:

Proof:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Does it work for $n=1$?

$$a_0 = \frac{f(1) + f(-1)}{2}$$

$$a_1 = \frac{f(1) - f(-1)}{2}$$

YES!

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f(1) \\ f(-1) \end{bmatrix}$$

Plan:
Use induction

Polynomial Representation Theorem

Proof

$$f_1$$

$$x_2 f_1$$

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

$$\begin{array}{lcl} f(1, 1) = a_{00} + a_{10} + a_{01} + a_{11} & \left. \vphantom{f(1, 1)} \right\} & x_2 = 1 \\ f(-1, 1) = a_{00} - a_{10} + a_{01} - a_{11} & \left. \vphantom{f(-1, 1)} \right\} & \\ f(1, -1) = a_{00} + a_{10} - a_{01} - a_{11} & \left. \vphantom{f(1, -1)} \right\} & x_2 = -1 \\ f(-1, -1) = a_{00} - a_{10} - a_{01} + a_{11} & \left. \vphantom{f(-1, -1)} \right\} & \end{array}$$

the only quadrant that changes sign

Polynomial Representation Theorem

Observation

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

$$\begin{aligned} f(1, 1) &= a_{00} + a_{10} + a_{01} + a_{11} \\ f(-1, 1) &= a_{00} - a_{10} + a_{01} - a_{11} \\ f(1, -1) &= a_{00} + a_{10} - a_{01} - a_{11} \\ f(-1, -1) &= a_{00} - a_{10} - a_{01} + a_{11} \end{aligned}$$

Observation: the signs in each column are the XORs of the corresponding variables

Polynomial Representation Theorem

Proof

The proof is by **induction**

We proved the base case: $n=1$ as well as $n=2$

Assume that:

$$\underline{f}_n = H_{2^n} A_n(f)$$

Need to prove that:

$$x_{n+1} = 1$$

$$\underline{f}_{n+1} =$$

$$x_{n+1} = -1$$

$$\begin{bmatrix} H_{2^n} \\ H_{2^n} \end{bmatrix} \begin{bmatrix} H_{2^n} \\ -H_{2^n} \end{bmatrix} A_{n+1}(f)$$

$$\underline{f}_n = H_{2^n} A_n(f)$$

Q

Part of the polynomial
without x_{n+1}

Part of the polynomial
with x_{n+1}

However we need to prove that:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f}_n$$

Polynomial Representation Theorem

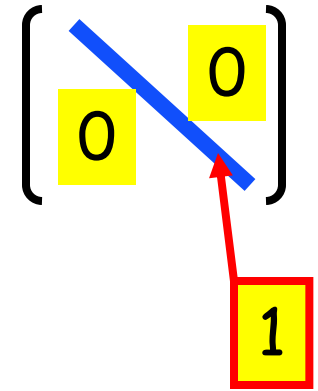
Proof

Lemma:

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

Where I_{2^n} is the identity matrix of dimension 2^n

Proof:



BY INDUCTION

Polynomial Representation Theorem

Putting it all together

We proved that:

$$\underline{f_n} = H_{2^n} A_n(f)$$

And also (Lemma):

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

Need to proved that:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Multiply both
Sides by:

$$H_{2^n} \cdot$$

$$\underline{f_n} = H_{2^n} A_n(f)$$

$$H_{2^n} \underline{f_n} = H_{2^n} \cdot H_{2^n} A_n(f)$$

By the lemma:

$$H_{2^n} \underline{f_n} = 2^n I_{2^n} A_n(f)$$

Q

Polynomial Representation Theorem

Representation Theorem:

(Fourier 18xx, Plancherel 19xx, Muller 195x, Ninomiya 195x)

- (i) Every Boolean function has a unique representation as a multilinear polynomial of the form:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

- (ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

P2. Zero-Value:

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha} = f(1, 1, \dots, 1) \in \{1, -1\}$$

P3. Moments:

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$$

P4. Correlation:

$$\sum_{X \in \{1,-1\}^n} f(X) \hat{f}(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \hat{a}_{\alpha}$$

P5. Parseval's Property:

$$\sum_{X \in \{1,-1\}^n} f^2(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 2^n$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

Proof:

BY INDUCTION

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

Proof: X^{α} is a row in the matrix

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

The all-1 row balances because of the negation

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

Proof:

Every row but the first is balanced (same number of 1's and -1's).

Also,

X^{α} is an XOR function of $|\alpha|$ variables...

Q

Important note: all terms vanish but the constant term

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P2. Zero-Value:

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha} = f(1, 1, \dots, 1) \in \{1, -1\}$$

Proof:

Let $X = (1, 1, \dots, 1)$ in $f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$

Q

Notice that:

$$\left| \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \right| = 1$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

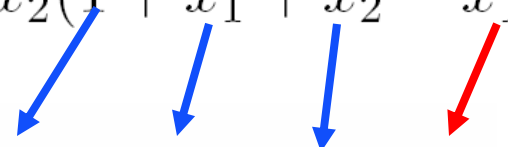
P3. Moments:

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$$

Proof: $X^{\alpha_1} X^{\alpha_2} = 1$ if and only if $\alpha_1 = \alpha_2$

The constant term in $f(X) X^{\alpha}$ is a_{α}

Example: $AND(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1 x_2)$

$$\begin{aligned} x_1 x_2 AND(x_1, x_2) &= \frac{1}{2} x_1 x_2 (1 + x_1 + x_2 - x_1 x_2) \\ &= \frac{1}{2} (x_1 x_2 + x_2 + x_1 - 1) \end{aligned}$$


Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P3. Moments:

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$$

Proof: $X^{\alpha_1} X^{\alpha_2} = 1$ if and only if $\alpha_1 = \alpha_2$

The constant term in $f(X) X^{\alpha}$ is a_{α}

By P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n \times \text{constant term} = 2^n a_{\alpha}$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P4. Correlation:

$$\sum_{X \in \{1,-1\}^n} f(X) \hat{f}(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \hat{a}_{\alpha}$$

Proof: $\hat{f}(X) = \sum_{\alpha \in \{0,1\}^n} \hat{a}_{\alpha} X^{\alpha}$

Repeated application of:

P3. Moments: $\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P5. Parseval's Property:

$$\sum_{X \in \{1,-1\}^n} f^2(X) \stackrel{?}{=} 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 2^n$$

Proof:

application of

P4. Correlation:
$$\sum_{X \in \{1,-1\}^n} f(X) \hat{f}(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \hat{a}_{\alpha}$$

With
$$f(X) \equiv \hat{f}(X)$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P5. Parseval's Property:

$$\sum_{X \in \{1,-1\}^n} f^2(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 \stackrel{?}{=} 2^n$$

Proof:

Remember that

$$f(X) \in \{1, -1\} \longrightarrow f^2(X) = 1$$

Q

The power spectrum of a Boolean function sums to 1

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 1$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P5. Parseval's Property:

$$\sum_{X \in \{1,-1\}^n} f^2(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 2^n$$

Example:

$$OR(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1 x_2)$$

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 1$$

$$1/4 + 1/4 + 1/4 + 1/4 = 1$$

Polynomial Representation

Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

P1. Orthogonality:

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

P2. Zero-Value:

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha} = f(1, 1, \dots, 1) \in \{1, -1\}$$

P3. Moments:

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$$

P4. Correlation:

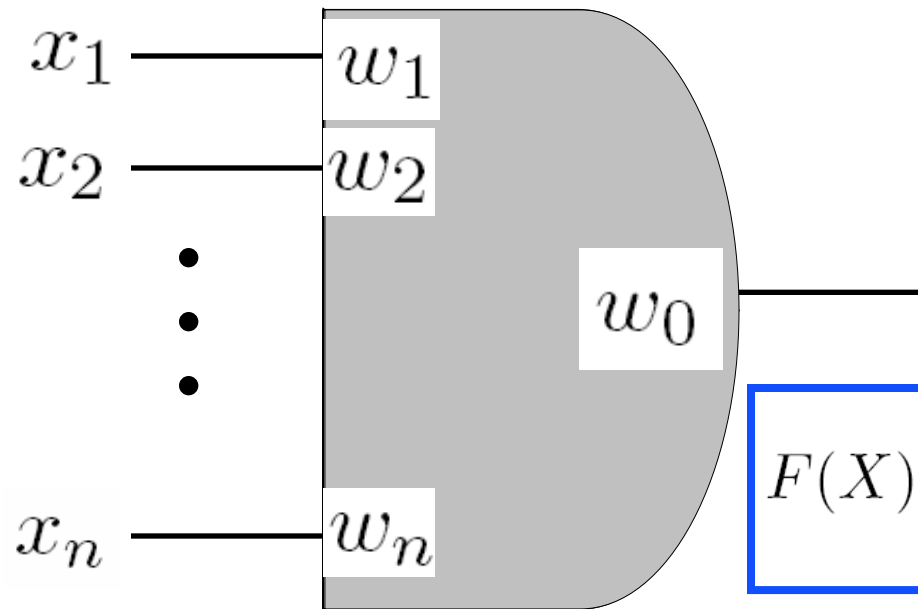
$$\sum_{X \in \{1,-1\}^n} f(X) \hat{f}(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \hat{a}_{\alpha}$$

P5. Parseval's Property:

$$\sum_{X \in \{1,-1\}^n} f^2(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 2^n$$

Back to Linear Threshold

Linear Threshold (LT) gate

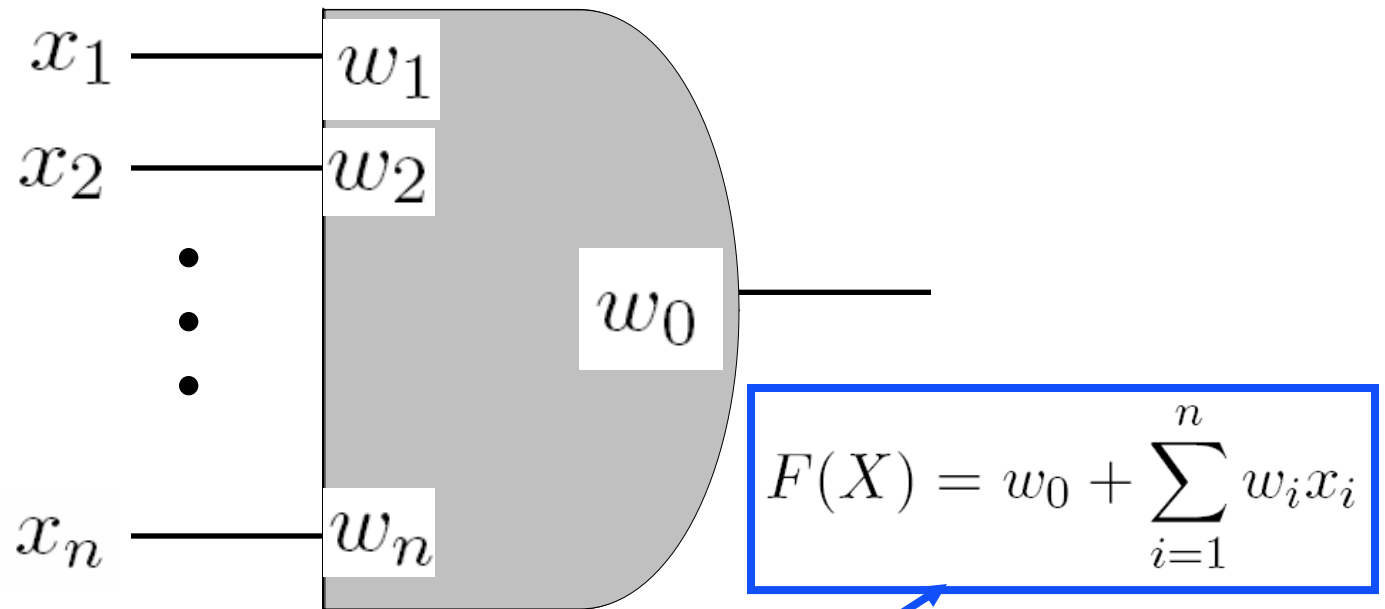


$$F(X) = w_0 + \sum_{i=1}^n w_i x_i$$

$$f(X) = \text{sgn}(F(X)) = \begin{cases} 1 & \text{if } F(X) < 0 \\ -1 & \text{if } F(X) > 0 \end{cases}$$

Back to Linear Threshold

Linear Threshold (LT) gate



$F(X)$ is a linear polynomial, generalization??

S-Threshold Functions

Definition: Let $S \subseteq \{0, 1\}^n$ be an arbitrary subset of binary vectors.

A **Boolean function** is an **S-threshold** function if there is a set of weight such that:

$$f(X) = \text{sgn}(F(X)) = \begin{cases} 1 & \text{if } F(X) > 0 \\ -1 & \text{if } F(X) < 0 \end{cases}$$

Where,

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

WLOG, can always
Adjust the constant

S-Threshold Functions

Examples

$$f(X) = \operatorname{sgn}(F(X)) = \begin{cases} 1 & \text{if } F(X) > 0 \\ -1 & \text{if } F(X) < 0 \end{cases}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

S is the set of binary vectors of **weight at most 1**:

$$S = \{0000, 0001, 0010, 0100, 1000\}$$

$$F(X) = w_{0000} + w_{1000}x_1 + w_{0100}x_2 + w_{0010}x_3 + w_{0001}x_4$$

Those are LT functions

S-Threshold Functions

Examples

$$f(X) = \operatorname{sgn}(F(X)) = \begin{cases} 1 & \text{if } F(X) > 0 \\ -1 & \text{if } F(X) < 0 \end{cases}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

S is the set of binary vectors of weight at most 2:

Those are Quadratic Threshold functions

S is the set of binary vectors of weight at most n :

Those are all possible Boolean functions...

Same as polynomial representation, we do not need the sgn ...

S-Threshold Functions

Spectral Characterization Theorem

Theorem (Characterization of S-Threshold): C-Theorem

Let f be Boolean function and let $S \subseteq \{0, 1\}^n$
be an arbitrary subset of binary vectors

Let $f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$ and $F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$

Then (the claim):

If and only if

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

S-Threshold Functions

Spectral Characterization Theorem

Theorem (Characterization of S-Threshold): C-Theorem

Let f be Boolean function and let $S \subseteq \{0, 1\}^n$
be an arbitrary subset of binary vectors

Let $f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$ and $F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$

Then (the claim):

Many local conditions

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

A global condition

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Spectral Characterization Theorem

Example

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

$$AND(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1 x_2)$$

$$OR(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1 x_2)$$

$$F(X) = -1 + x_1 + x_2$$

$$2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha} =$$

$$4((-1) \times \frac{1}{2} + 1 \times \frac{1}{2} + 1 \times \frac{1}{2})$$

$$= 2$$

$$4((-1) \times (-\frac{1}{2}) + 1 \times \frac{1}{2} + 1 \times \frac{1}{2})$$

$$= 6$$

$$\sum_X |F(X)| = 1 + 1 + 1 + 3 = 6$$

Spectral Characterization Theorem

Example

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

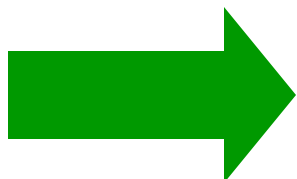
$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

$$OR(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1 x_2)$$

$$F(X) = -1 + x_1 + x_2$$

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

$x_1 x_2$	$\downarrow$
11	1
1-1	-1
-11	-1
-1-1	-3



$$OR(x_1, x_2) = \text{sgn}(-1 + x_1 + x_2)$$

Spectral Characterization Theorem

Observation

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

In the AND/OR example we looked only at **three** spectral coefficients (out of **four**)

In general, to recognize an **S-threshold** we need to consider only **|S|** spectral coefficients

C -Thm:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha} \quad F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

if and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Application: XOR is not Linear Threshold

Proof: By contradiction, assume $XOR(x_1, x_2, \dots, x_n) = \text{sgn}(\sum_{\alpha \in S} w_{\alpha} X^{\alpha})$

vectors of weight at most 1

By the C-Thm:

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

$F(X)$

However, $XOR(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$

= ??

$\Rightarrow a_{\alpha} = 1$ for $\alpha = 111 \dots 1$ and $a_{\alpha} = 0$ otherwise

$$\sum_{X \in \{1, -1\}^n} |F(X)|$$

= 0

$$F(X) \equiv 0$$

$$XOR(X) = \text{sgn}(F(X)) \equiv 1$$

Contradiction

Q

C -Thm:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha} \quad F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

if and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Generalization: XOR is not S-Threshold if $(1, 1, \dots, 1, 1) \notin S$

Proof: By contradiction, assume $XOR(x_1, x_2, \dots, x_n) = \text{sgn}(\sum_{\alpha \in S} w_{\alpha} X^{\alpha})$

$$(1, 1, \dots, 1, 1) \notin S$$

By the C-Thm:

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

$$F(X)$$

However, $XOR(x_1, x_2, \dots, x_n) = x_1 x_2 \cdots x_n$

$\Rightarrow a_{\alpha} = 1$ for $\alpha = 111 \dots 1$ and $a_{\alpha} = 0$ otherwise

$$\sum_{X \in \{1, -1\}^n} |F(X)|$$

$$= 0$$

$$F(X) \equiv 0$$

Contradiction

$$\Rightarrow XOR(X) = \text{sgn}(F(X)) \equiv 1$$

Q

C -Thm:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha} \quad F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

if and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Sampling Theorem:

Let f_1 and f_2 be Boolean functions. Let f_1 be an S -threshold function for some given S . Then:

$$f_1(X) = f_2(X) \quad \forall X \in \{1, -1\}^n$$



Spectrum of f_1

Spectrum of f_2

$$a_{\alpha}^1 = a_{\alpha}^2 \quad \forall \alpha \in S$$

Every S -threshold function has a **unique spectrum on S**

C -Thm:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha} \quad F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

if and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Sampling Theorem: Let f_1 and f_2 be Boolean functions. Let f_1 be an S -threshold function for some given S . Then:

$$f_1(X) = f_2(X) \quad \forall X \in \{1, -1\}^n \iff a_{\alpha}^1 = a_{\alpha}^2 \quad \forall \alpha \in S$$

Proof:



Follows from the uniqueness of the spectrum

C -Thm:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha} \quad F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n \quad \text{if and only if}$$

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Sampling Theorem: Let f_1 and f_2 be Boolean functions. Let f_1 be an S -threshold function for some given S . Then:

$$f_1(X) = f_2(X) \quad \forall X \in \{1, -1\}^n \iff a_{\alpha}^1 = a_{\alpha}^2 \quad \forall \alpha \in S$$

Proof:

f_1 is S -threshold:

$$f_1(X) = \text{sgn}(F(X))$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

By C-Thm:

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}^1$$

$$a_{\alpha}^1 = a_{\alpha}^2 \quad \forall \alpha \in S$$

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}^2$$

By C-Thm:

$$f_2(X) = \text{sgn}(F(X))$$

$$f_1(X) = f_2(X)$$

Q

Sampling Theorem:

Let f_1 and f_2 be Boolean functions. Let f_1 be an S -threshold function for some given S . Then:

$$f_1(X) = f_2(X) \quad \forall X \in \{1, -1\}^n$$



$$a_{\alpha}^1 = a_{\alpha}^2 \quad \forall \alpha \in S$$

Counting Theorem:

For a given $S \subseteq \{0, 1\}^n$
the number of S -threshold functions $\leq 2^{O(n \cdot |S|)}$

QED
QED

Proof:

- Every S -threshold function has a unique spectrum on S
- How many different values can a spectral coefficient have?
- Inner product between two $(1, -1)$ vectors of length 2^n

Upper bound on the number
of different s -spectrums

$$(2^n + 1)^{|S|}$$

$$2^n + 1$$

Counting LT functions

Counting Theorem:

For a given $S \subseteq \{0, 1\}^n$
the number of S -threshold functions $\leq 2^{O(n \cdot |S|)}$

For LT functions: $|S| = n + 1$

Number of LT functions $\leq (2^n + 1)^{n+1} \leq 2^{(n+1)^2}$

It was proved (recently) that it is $\approx 2^{0.5n^2}$

Spectral Characterization Theorem

Proof Ideas

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Idea 1:

$$\sum_{X \in \{1, -1\}^n} f(X) F(X) = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Why?

The constant term in

$$f(X) F(X)$$

is

$$\sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Follows from:

P1. Orthogonality:

P4. Correlation:

Spectral Characterization Theorem

Proof Ideas

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \operatorname{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

Idea 2:

$$f(X) = \operatorname{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = \sum_{X \in \{1, -1\}^n} f(X) F(X)$$

$$\operatorname{sgn}(F(X)) \cdot F(X) = |F(X)|$$

$$f(X) = \operatorname{sgn}(F(X))$$

Spectral Characterization Theorem

C-Theorem:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

$$F(X) = \sum_{\alpha \in S} w_{\alpha} X^{\alpha}$$

$$f(X) = \text{sgn}(F(X)) \quad \forall X \in \{1, -1\}^n$$

If and only if

$$\sum_{X \in \{1, -1\}^n} |F(X)| = 2^n \sum_{\alpha \in S} w_{\alpha} a_{\alpha}$$

For more details see the paper on the class web site:

JB 1990, *"Harmonic Analysis of Polynomial Threshold Functions"*