

CNS 188a Overview

Emil Post
1897-1954



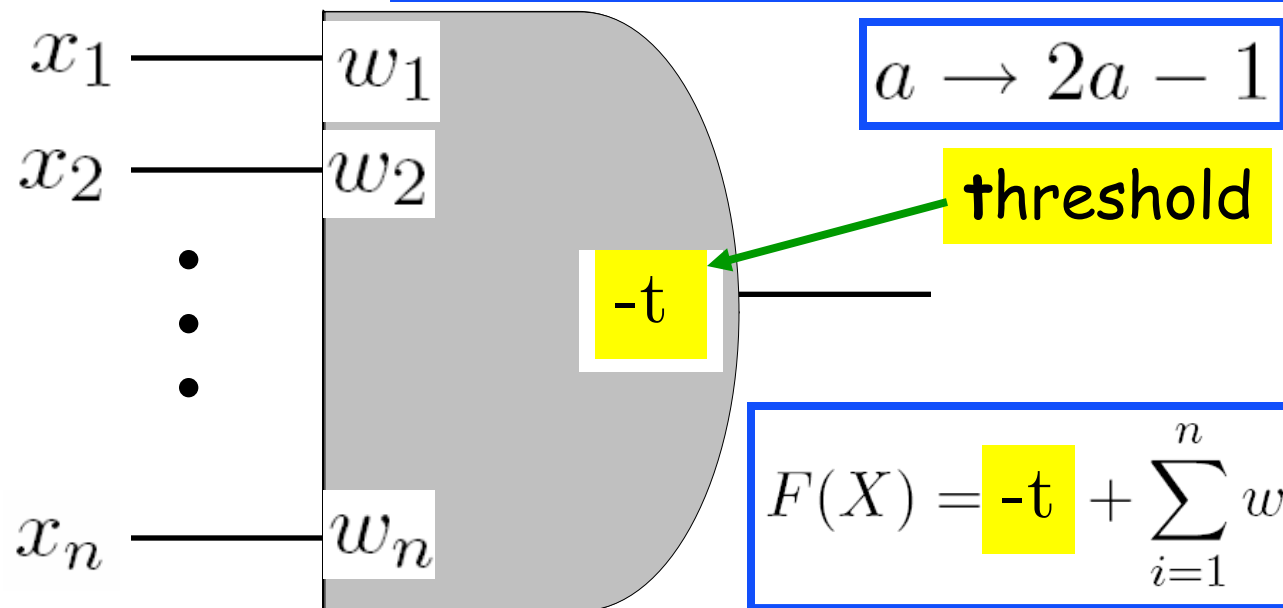
- Boolean algebra as an axiomatic system
- Boolean functions and their representations
Boolean formulas and spectral methods
- Implementing Boolean functions with relay circuits, circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits

Compositions of
Boolean functions
Universal Algebra

Linear Threshold Some Adjustments

Linear Threshold (LT) gate

$$f : \{-1, 1\}^n \longrightarrow \{-1, 1\}$$



$$f(X) = \operatorname{sgn}(F(X)) = \begin{cases} -1 & \text{if } F(X) < 0 \\ 1 & \text{if } F(X) \geq 0 \end{cases}$$

Linear Threshold Example

The **AND** function of two variables:

$$f(x_1, x_2) = \text{sgn}(-1 + x_1 + x_2)$$



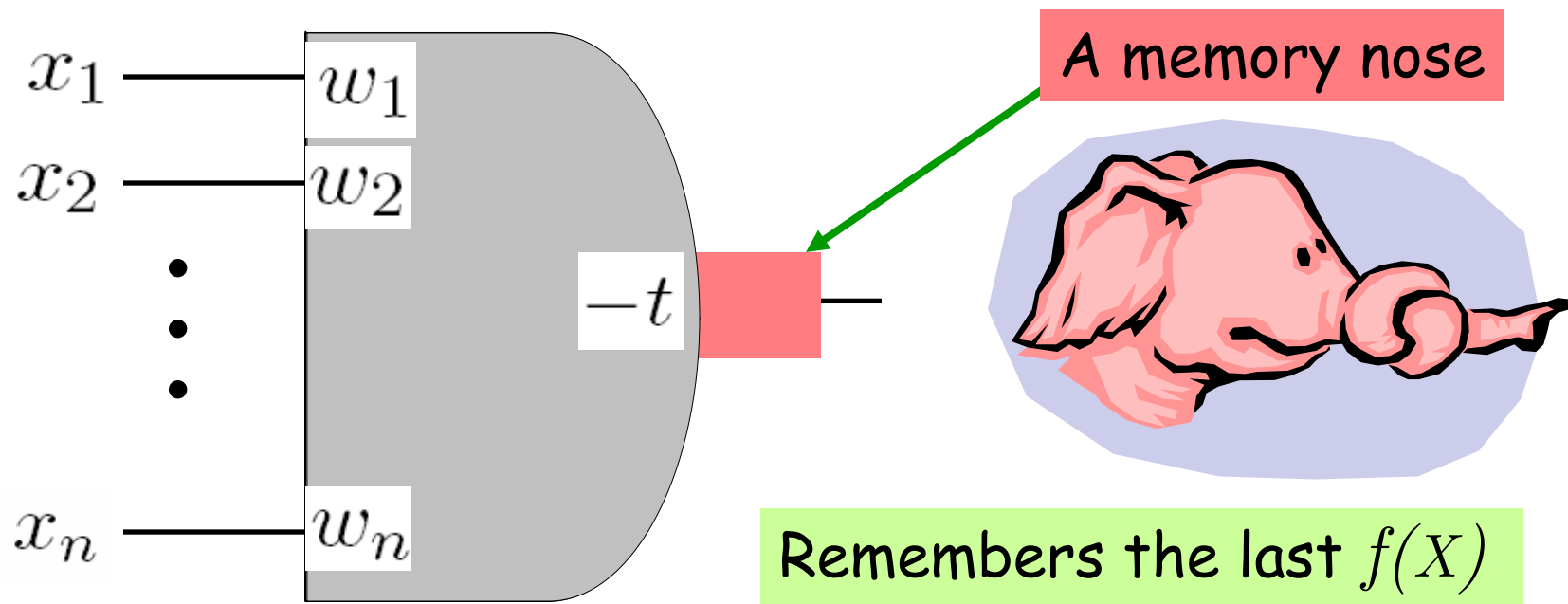
x_1x_2	
-1-1	-3
-11	-1
1-1	-1
11	1

Linear Threshold

New Ingredient

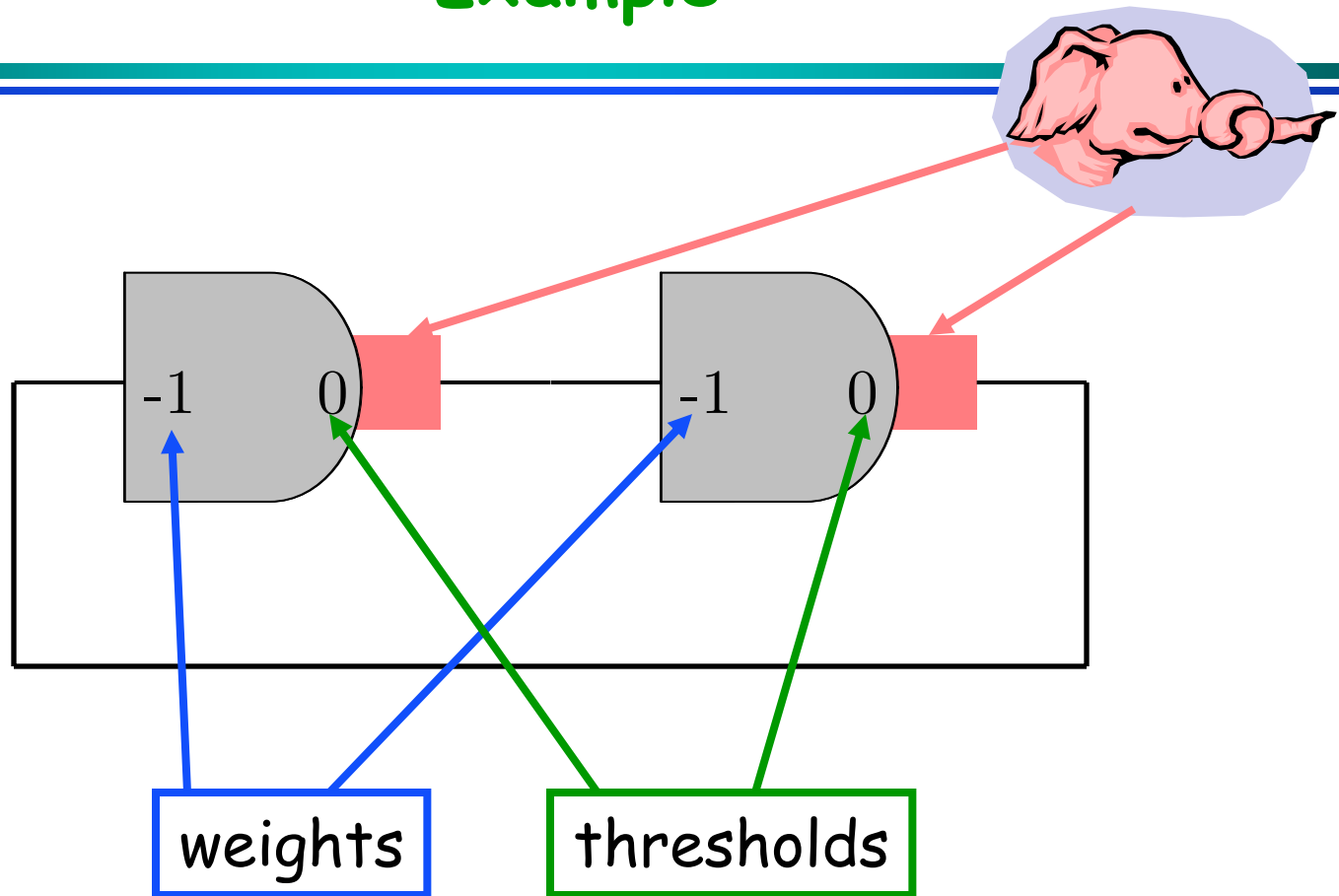
$$f(X) = \text{sgn}(F(X))$$

$$F(X) = -t + \sum_{i=1}^n w_i x_i$$



Feedback Networks

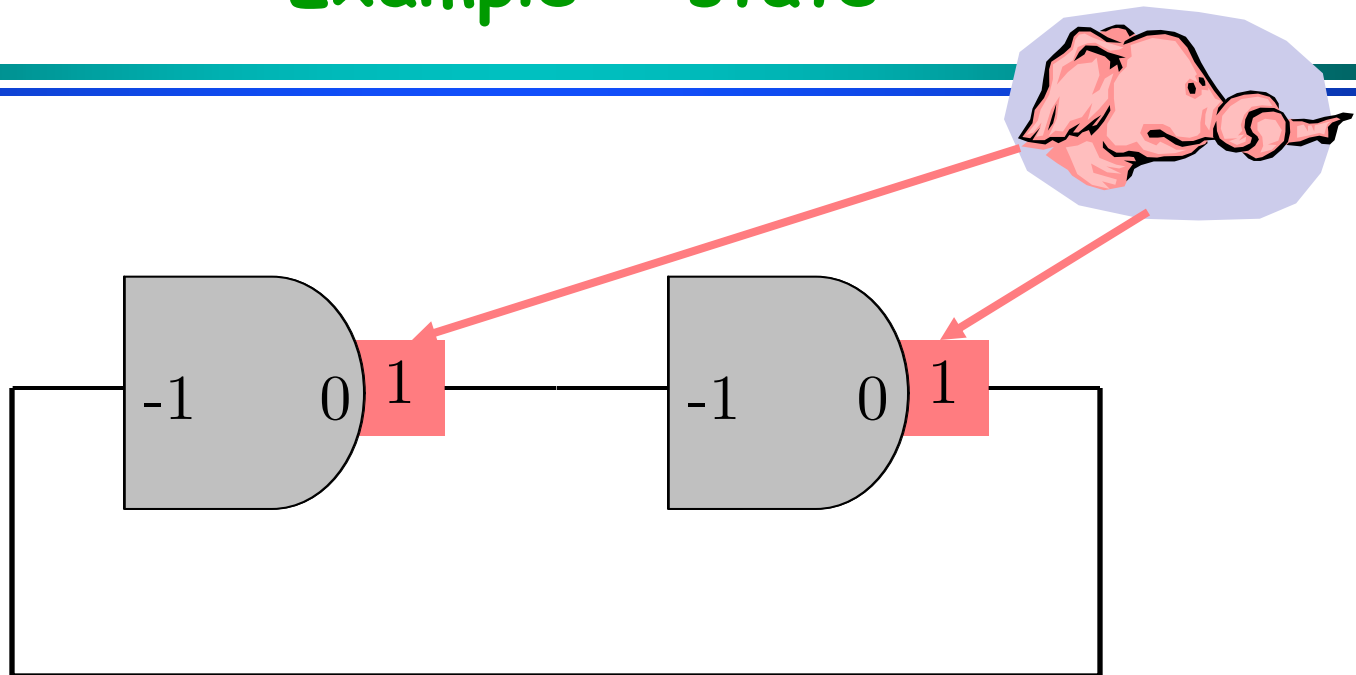
Example



The state of the network: the vector that corresponds to the states (noses...) of the gates

Feedback Networks

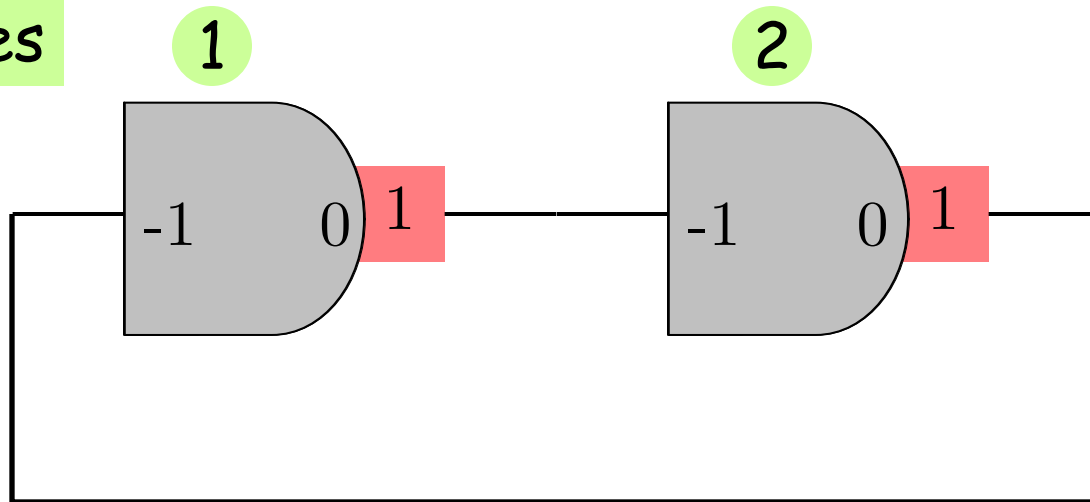
Example - state



Feedback Networks

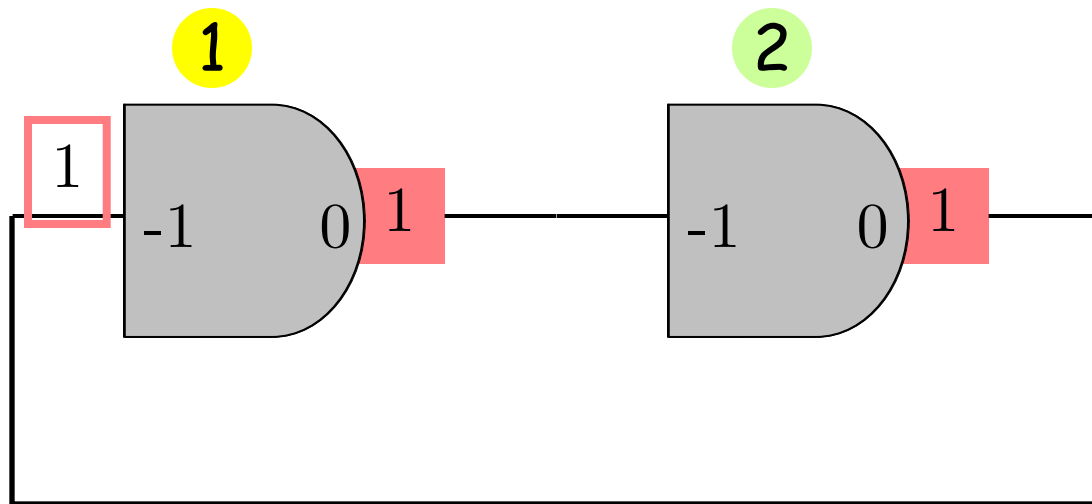
Example - gate numbering

Label the gates



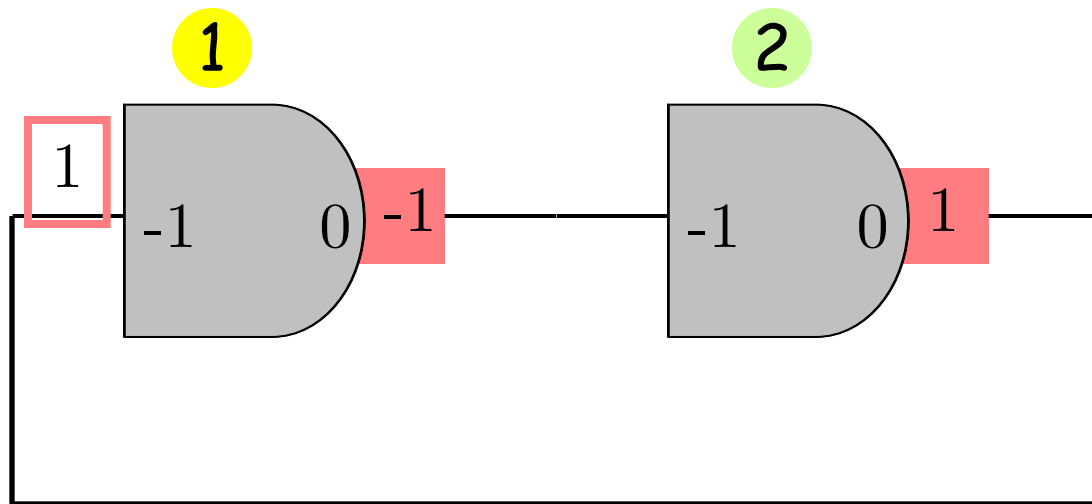
Feedback Networks

Example



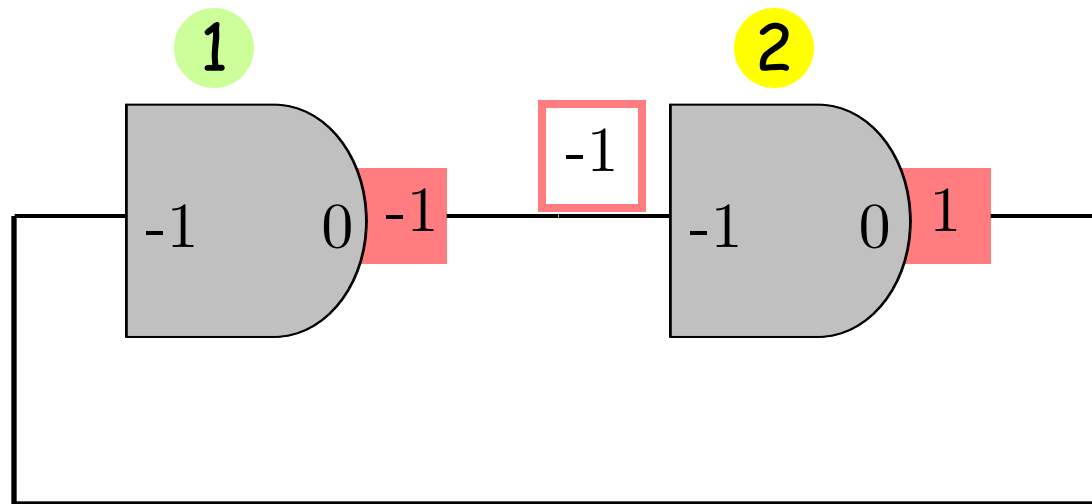
Feedback Networks

Example



Feedback Networks

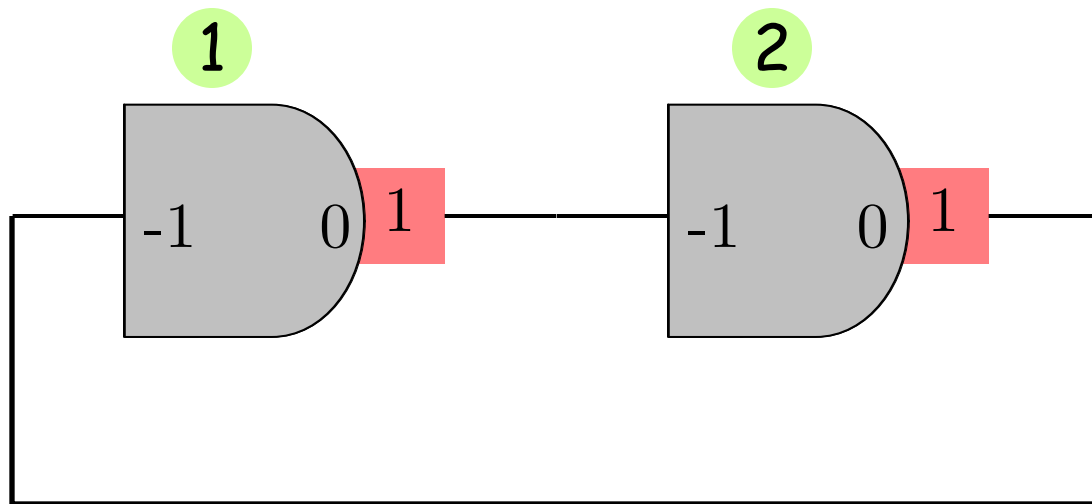
Example



-11 is a stable state

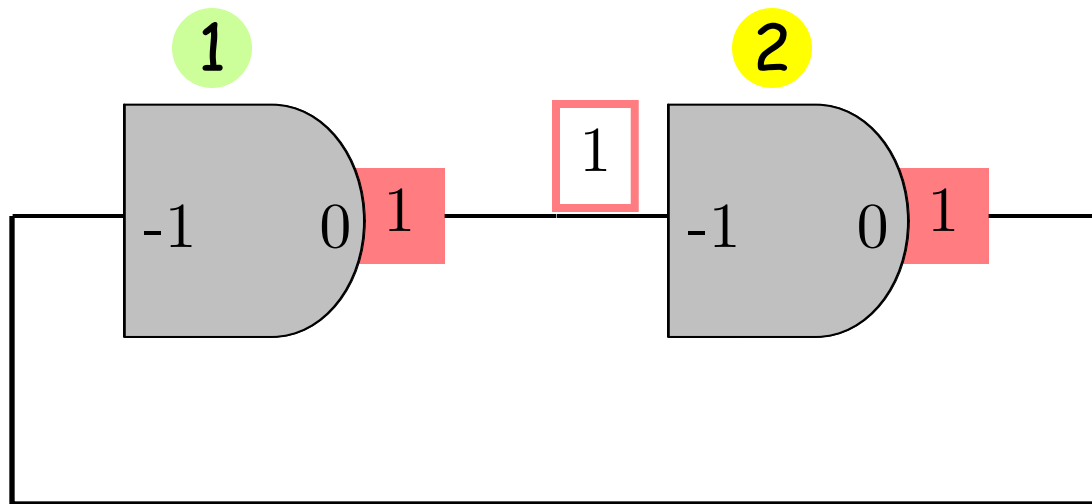
Feedback Networks

Example



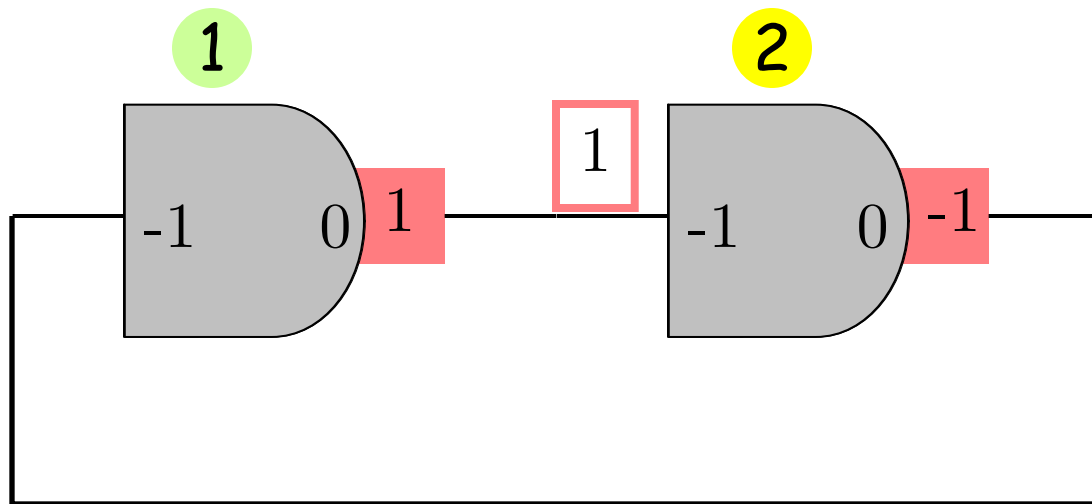
Feedback Networks

Example



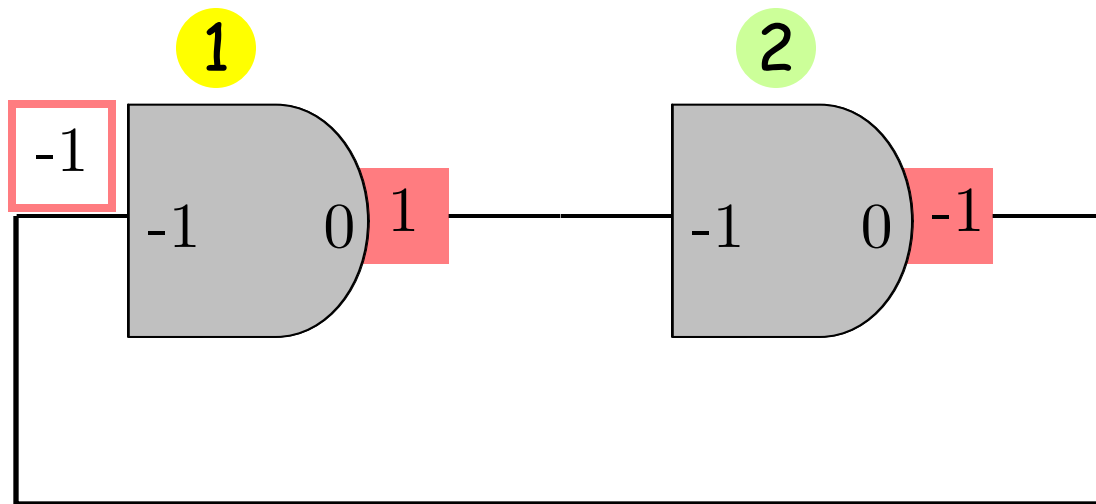
Feedback Networks

Example



Feedback Networks

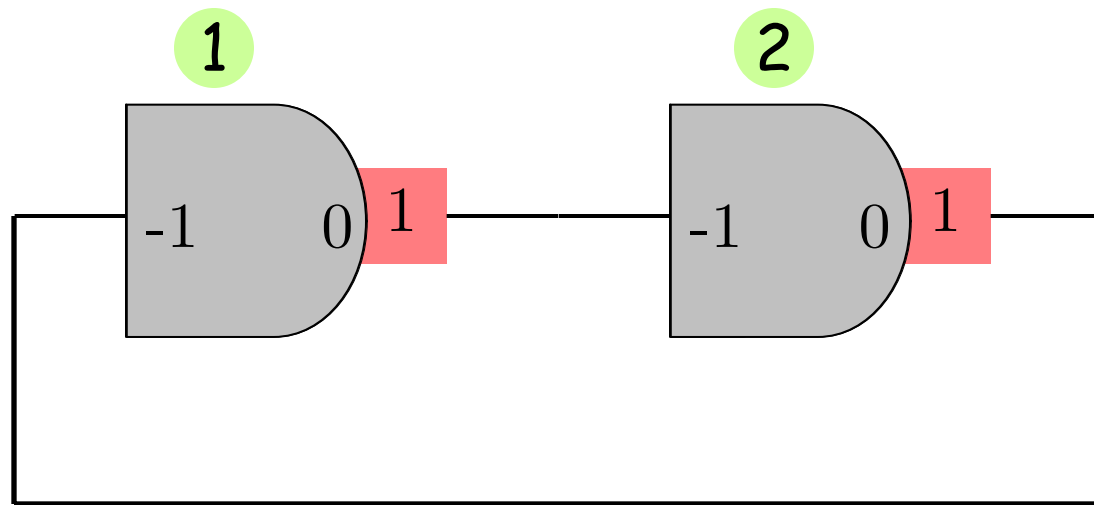
Example



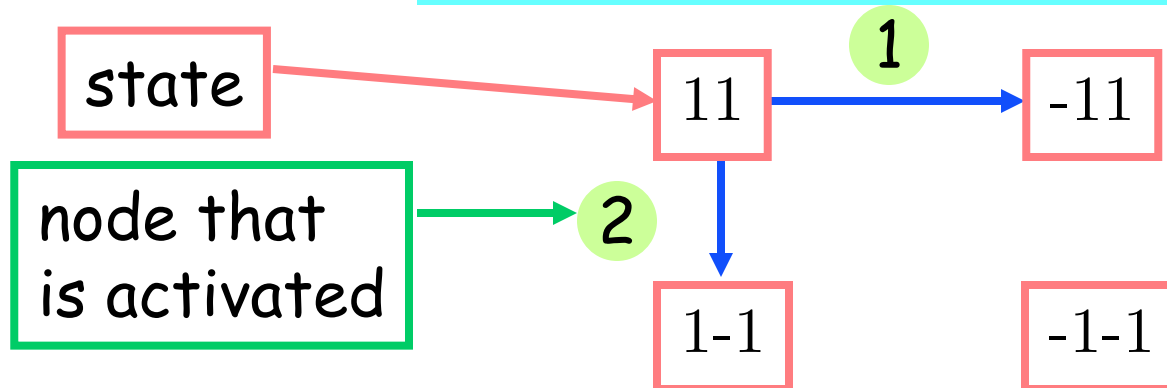
1-1 is a stable state

Feedback Networks

Example



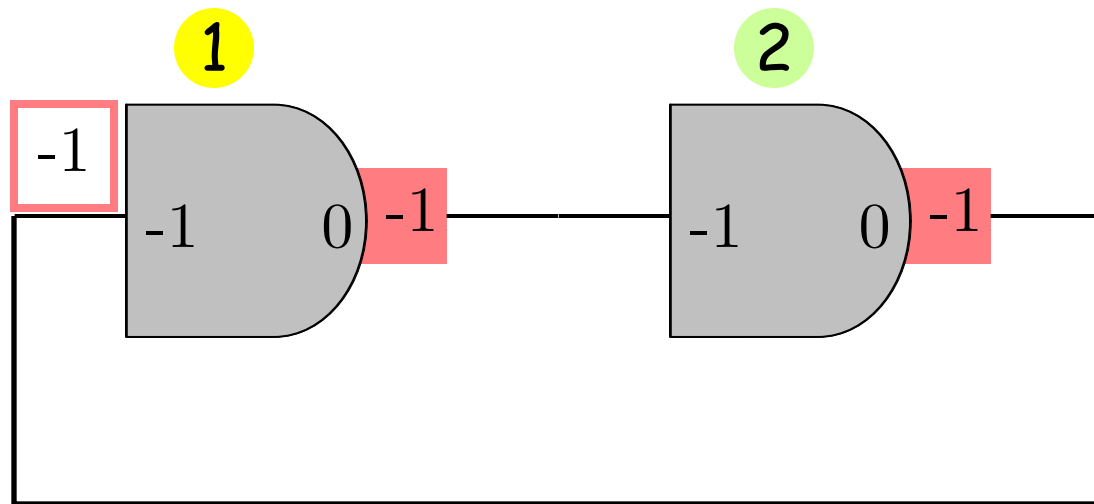
State transition diagram (state space)



Q: Is -1-1 a stable state?

Feedback Networks

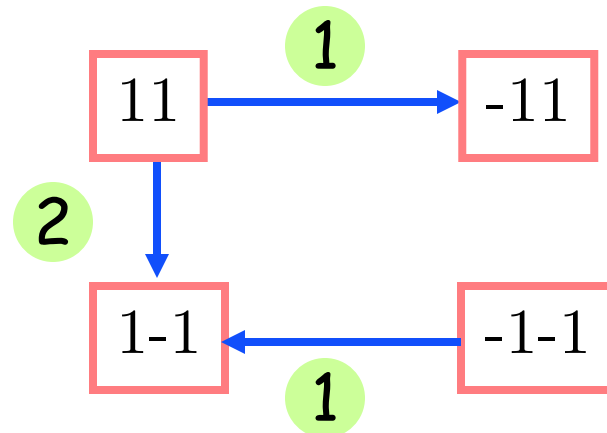
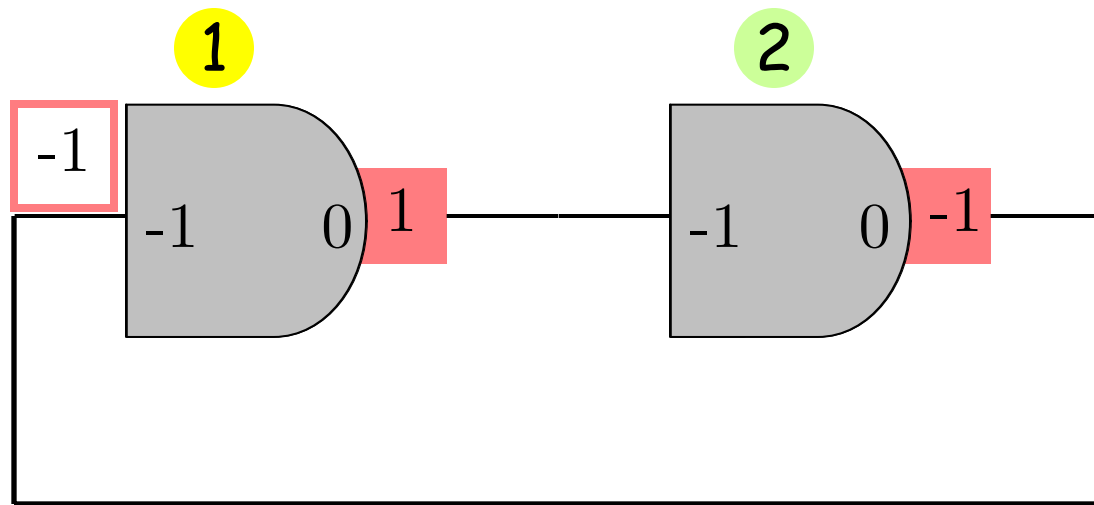
Example



Answer: No

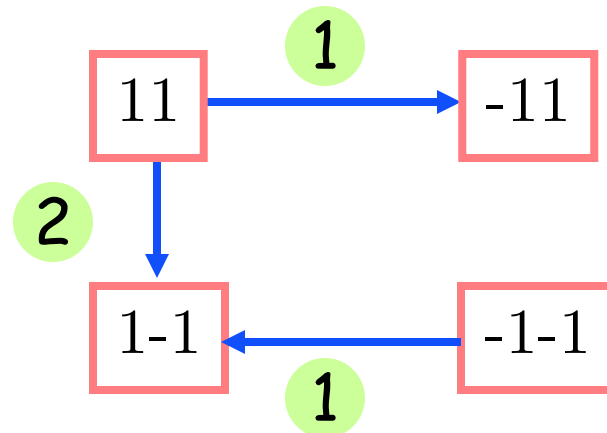
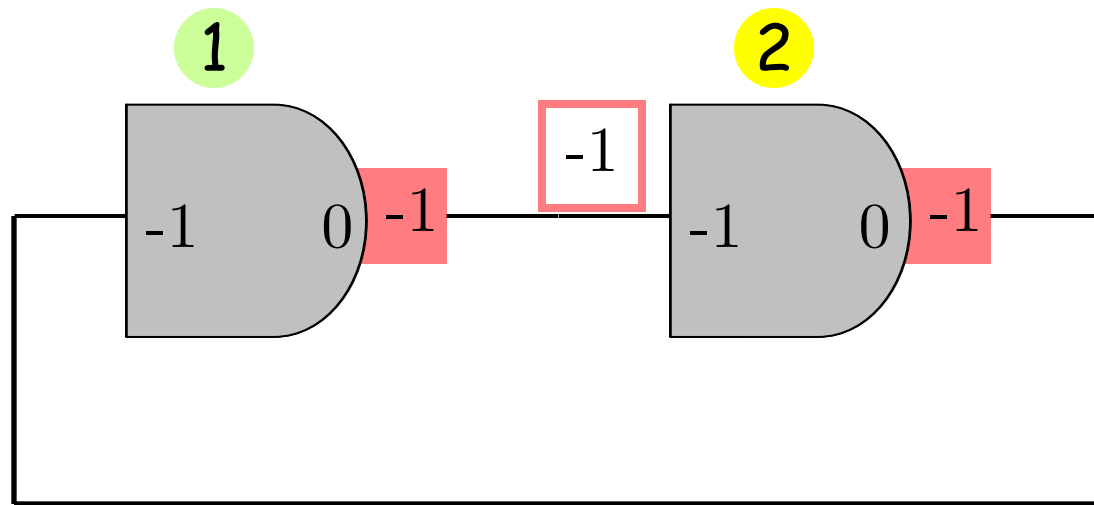
Feedback Networks

Example



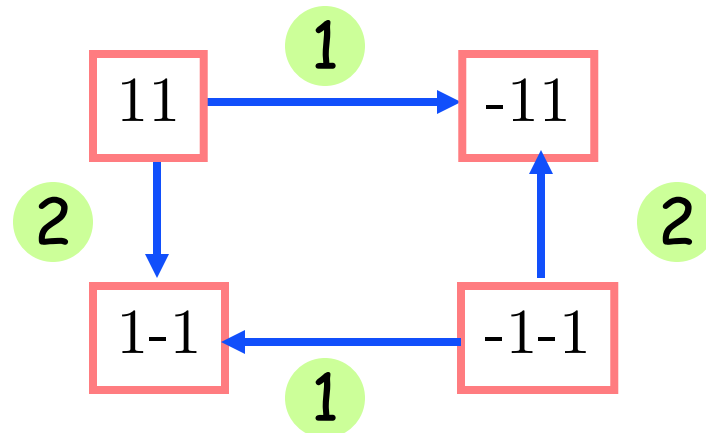
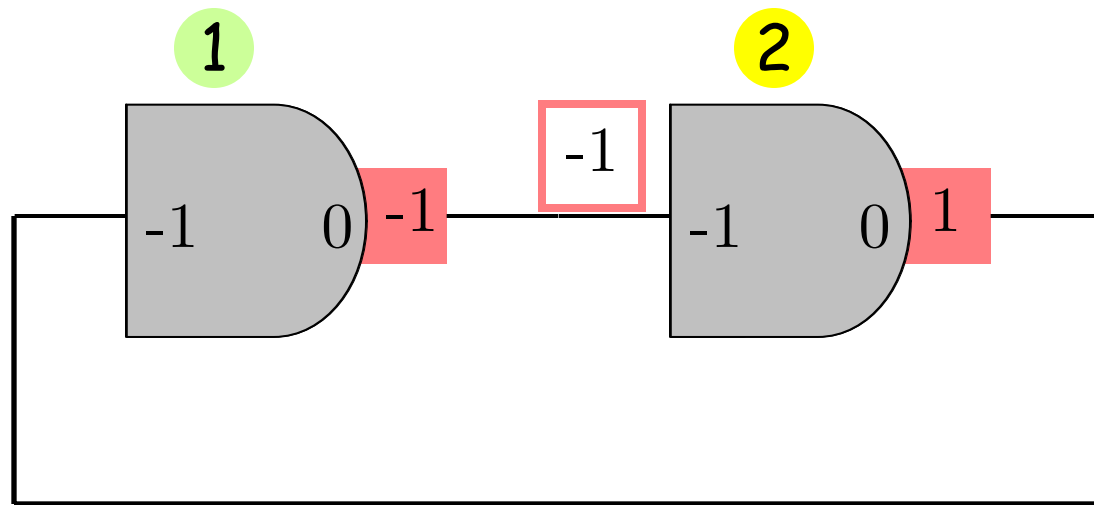
Feedback Networks

Example



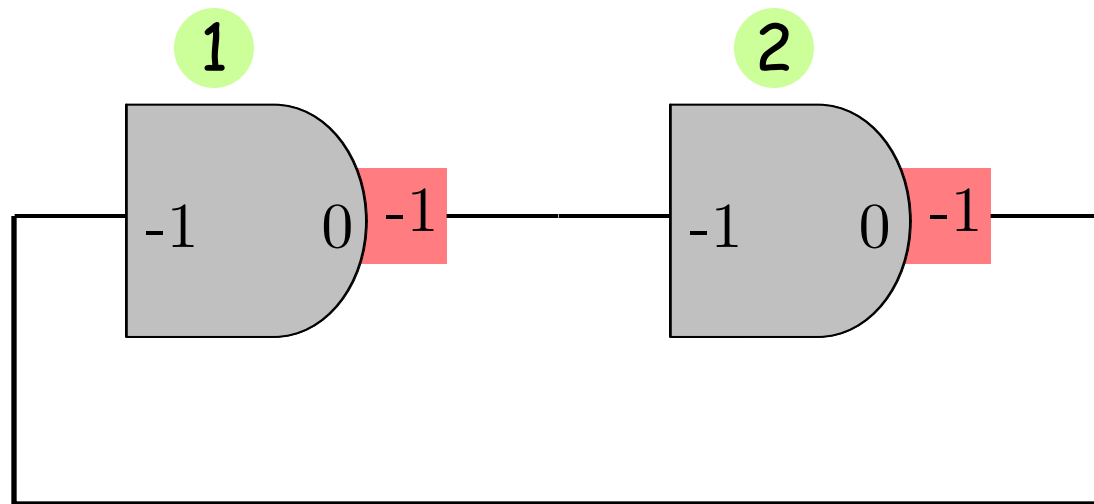
Feedback Networks

Example

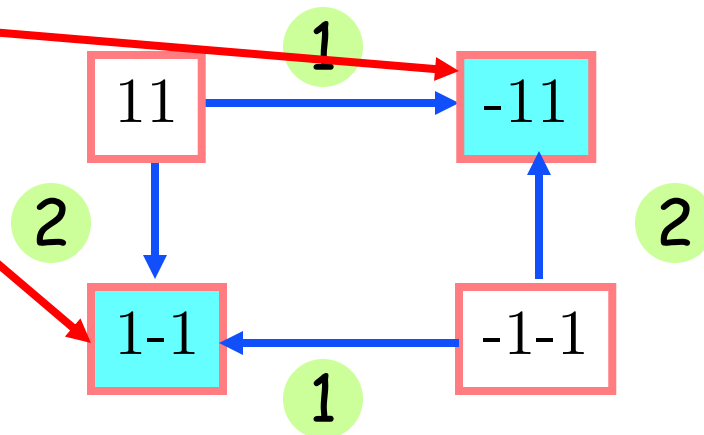


Feedback Networks

Example



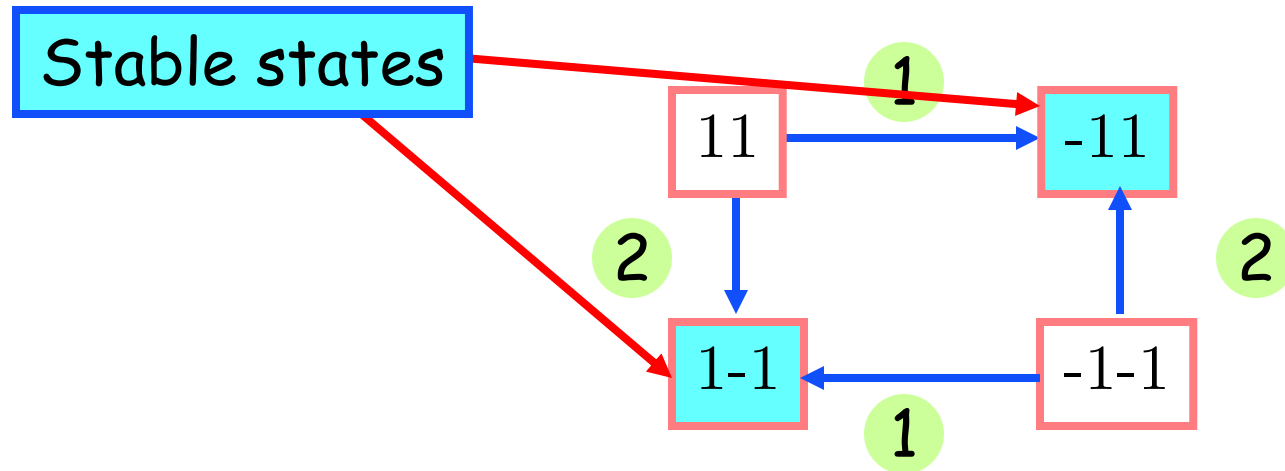
Stable states



Q:
why care about
stable states?

Feedback Networks

Computing with Dynamics



Input:
initial state

11

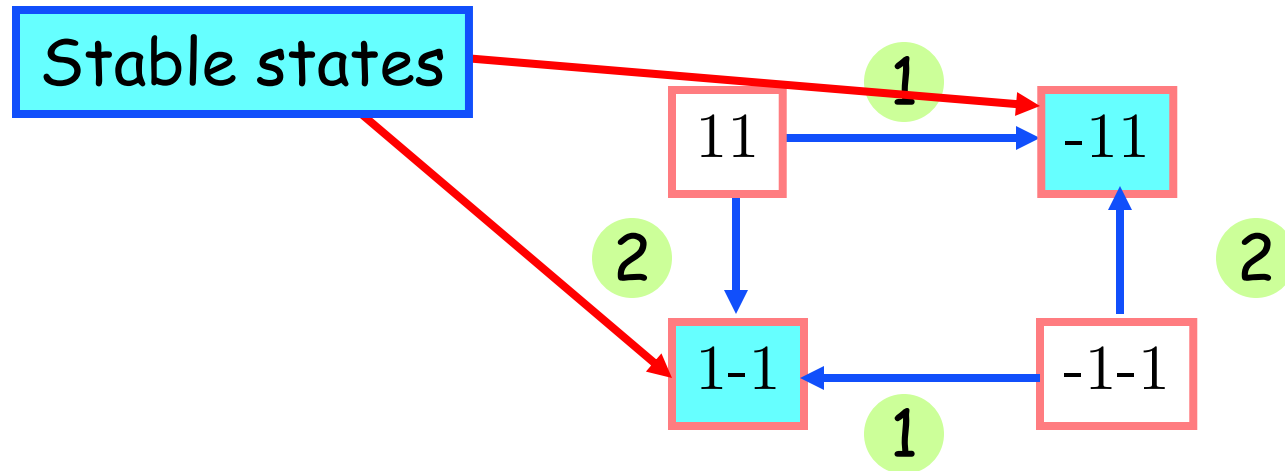
Feedback
Network

Output:
stable state

-11

Feedback Networks

Computing with Dynamics



Input:
initial state

11

Feedback
Network

Output:
stable state

1-1

Feedback Networks

Computing with Dynamics

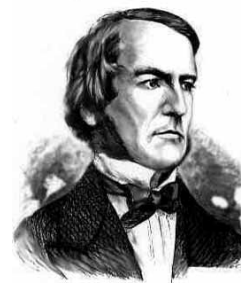
Associative Memory
"The Leibniz-Boole Machine"

Input:
initial state

Output:
stable state



Feedback
Network



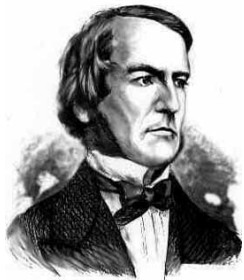
Feedback Networks

Computing with Dynamics

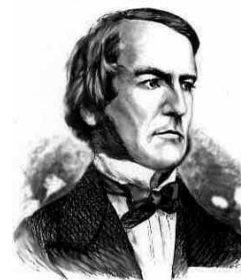
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Feedback
Network



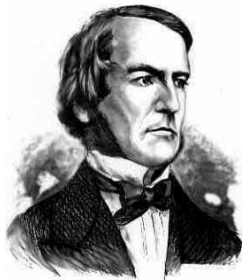
Feedback Networks

Computing with Dynamics

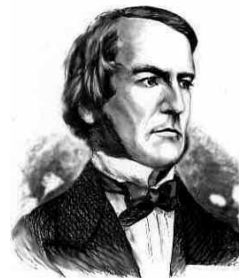
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Feedback
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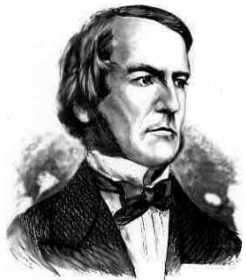
Feedback Networks

Computing with Dynamics

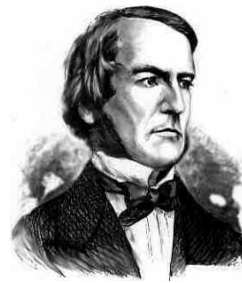
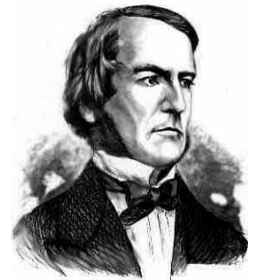
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Feedback
Network



Feedback Networks

Computing with Dynamics

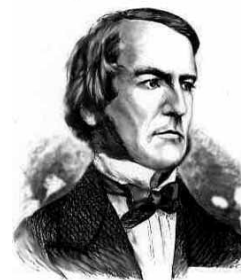
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Feedback
Network



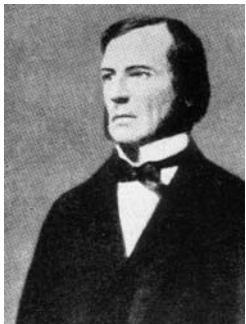
Feedback Networks

Computing with Dynamics

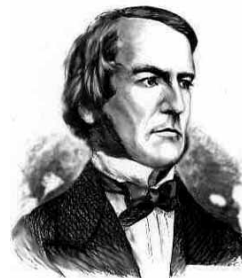
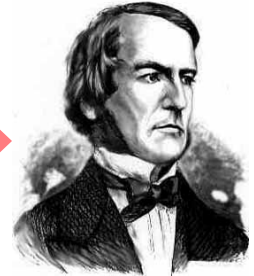
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Feedback
Network



Feedback Networks

Computing with Dynamics

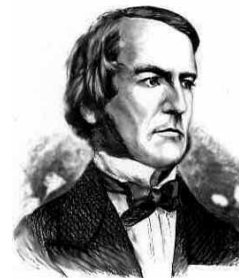
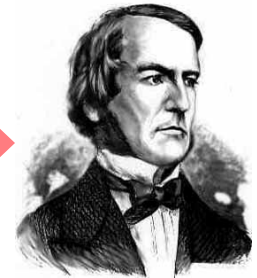
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Output:
stable state



Feedback
Network



Feedback Networks

Computing with Dynamics

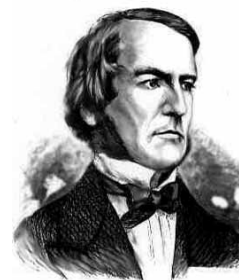
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Input:
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Output:
stable state



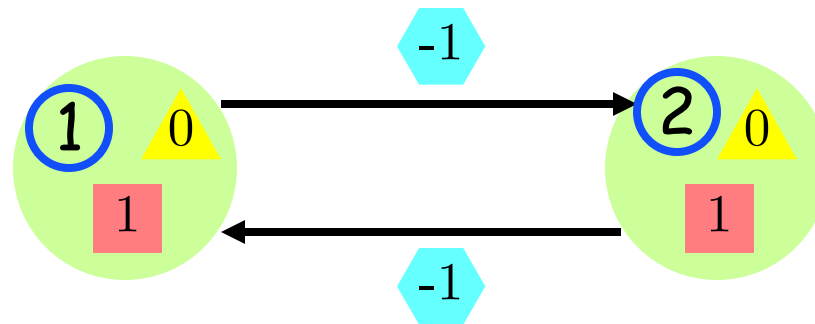
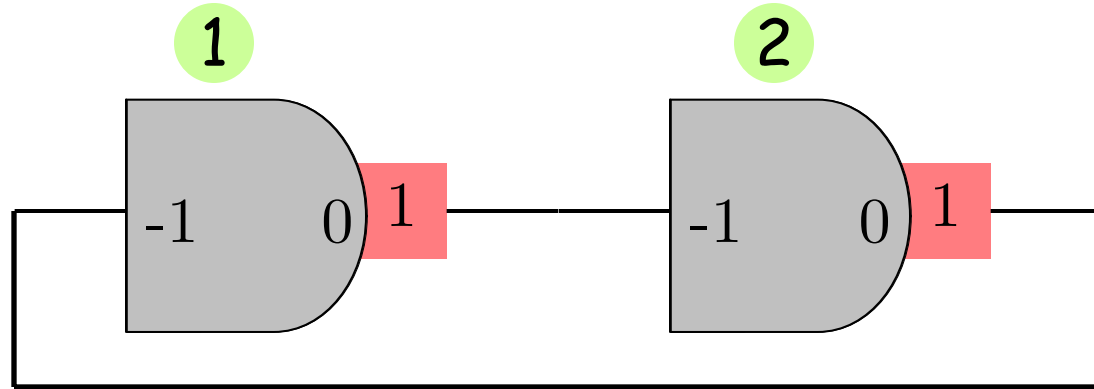
Feedback
Network





Feedback Networks

Hopfield Model (Caltech 1982)



$\textcircled{i}$ = node i

$\triangle 0$ = threshold t_i

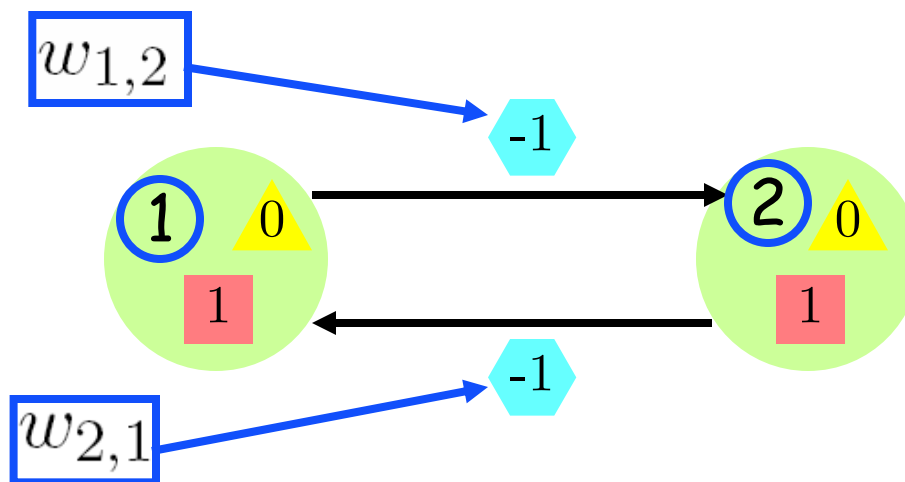
$\textcolor{red}{1}$ = state v_i

$\textcolor{cyan}{-1}$ = weight of edge (i,j)

Feedback Networks

Definition: feedback network

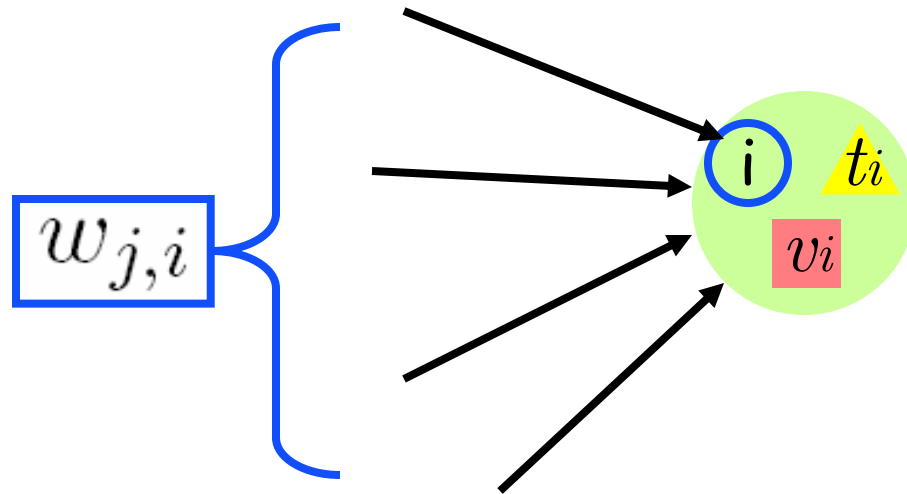
- A directed graph with nodes and directed edges
- Every node i has a state v_i
- Every node i has a threshold t_i
- Every directed edge (i,j) has a weight w_{ij}



Feedback Networks

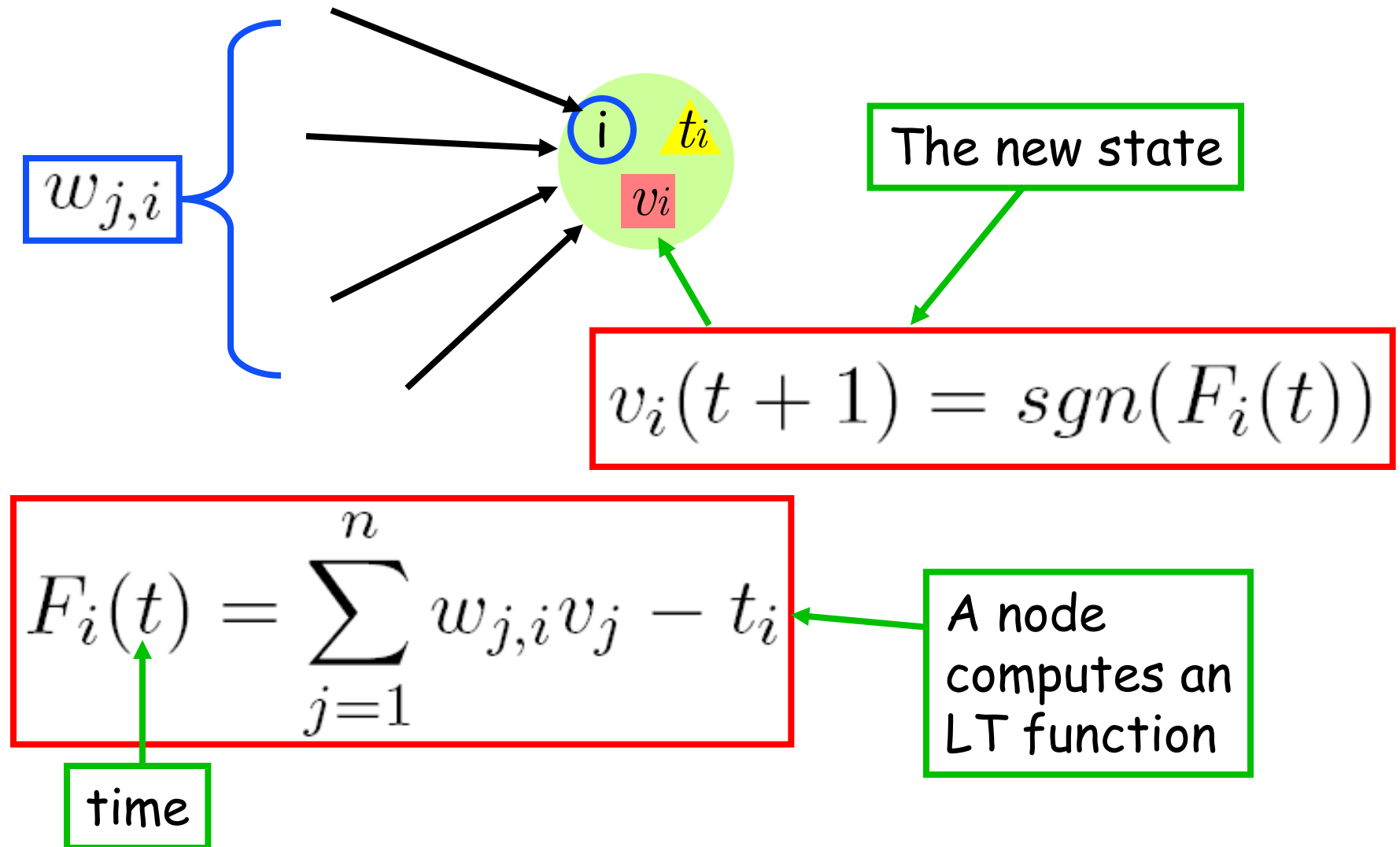
Definition: feedback network

- A directed graph with nodes and directed edges
- Every node i has a state v_i
- Every node i has a threshold t_i
- Every directed edge (i,j) has a weight w_{ij}



Feedback Networks

Computation in Each Node

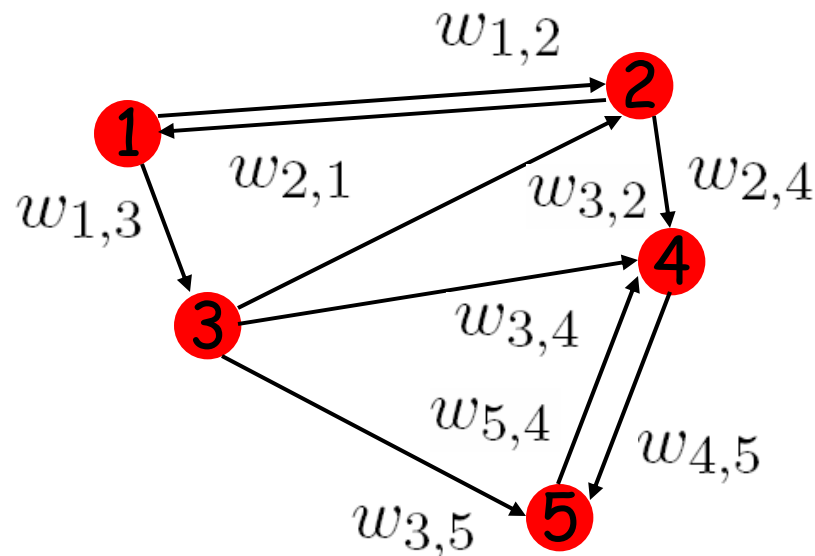


Feedback Networks

The Graph Description

Definition: feedback network

- A directed graph with nodes and directed edges
- Every node i has a state v_i
- Every node i has a threshold t_i
- Every directed edge (i,j) has a weight $w_{i,j}$



Feedback Networks

The Vector/Matrix Description

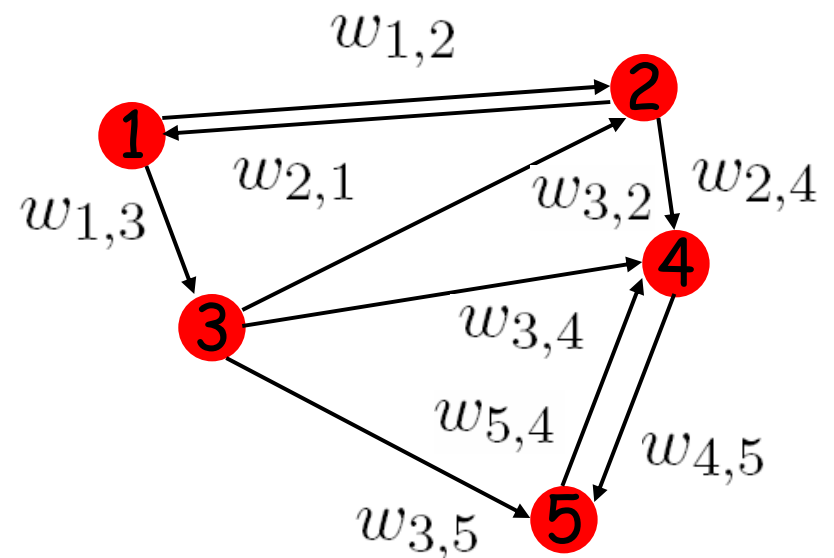
An n node feedback network can be specified by:

- W an $n \times n$ matrix of weights
- T an n vector of thresholds
- V an n vector of states

$$N = (W, T)$$

$$V = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

$$T = \begin{bmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ t_n \end{bmatrix}$$

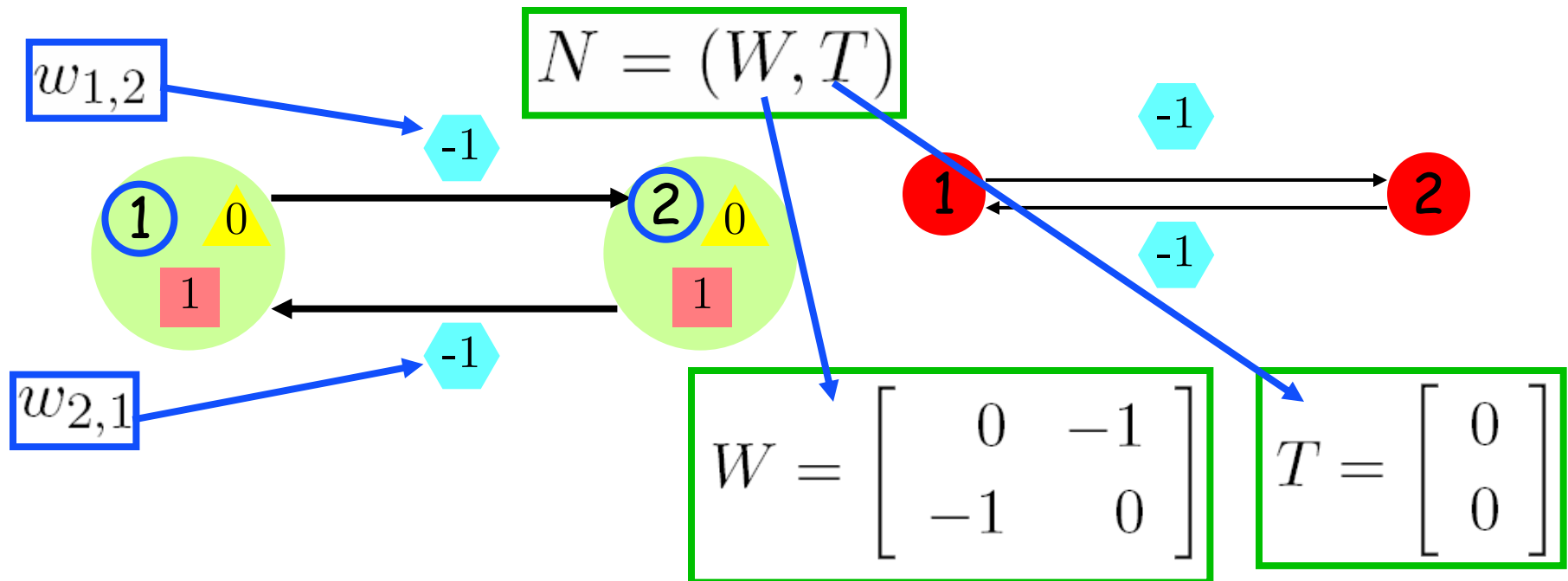


The Matrix Description

Example

An n node feedback network can be specified by:

- W an $n \times n$ matrix of weights
- T an n vector of thresholds
- V an n vector of states



The Matrix Description

Computation

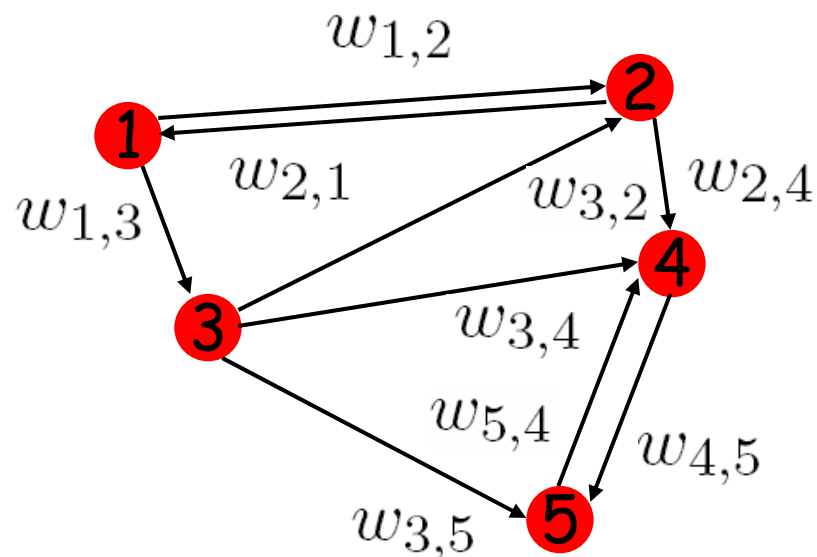
Computation in $N = (W, T)$

$$F_i(t) = \sum_{j=1}^n w_{j,i} v_j - t_i$$

by column

$$F(t) = W^T \cdot V - T$$

$$V(t+1) = \text{sgn}(F(t))$$

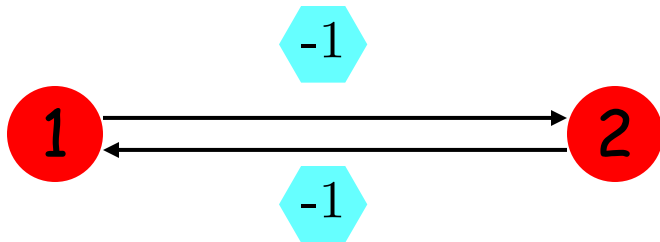


$$V = \begin{bmatrix} v_1 \\ v_2 \\ \cdot \\ \cdot \\ v_n \end{bmatrix}$$

$$T = \begin{bmatrix} t_1 \\ t_2 \\ \cdot \\ \cdot \\ t_n \end{bmatrix}$$

The Matrix Description

Example



$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$

$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Computation in $N = (W, T)$

$$F(t) = W^T \cdot V - T$$

$$\begin{bmatrix} F_1 \\ F_2 \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

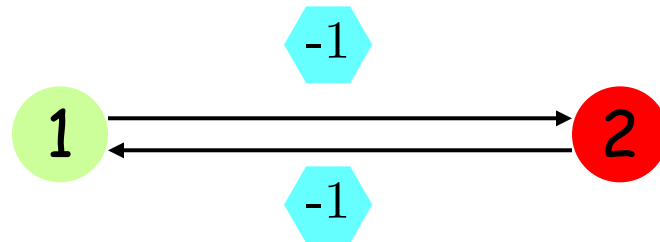
$$= \begin{bmatrix} -v_2 \\ -v_1 \end{bmatrix}$$

Q: when do the nodes compute?

Modes of Operation

Q: when do the nodes compute?

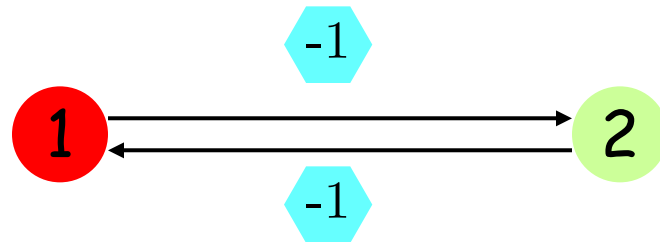
Serial mode: one node at a time (arbitrary order)



Modes of Operation

Q: when do the nodes compute?

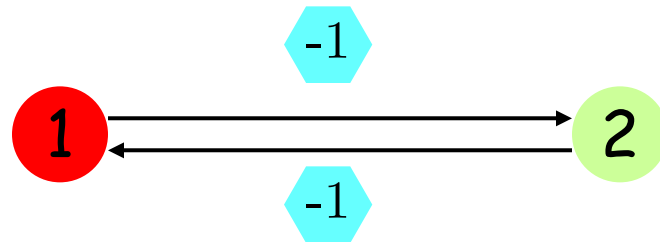
Serial mode: one node at a time (arbitrary order)



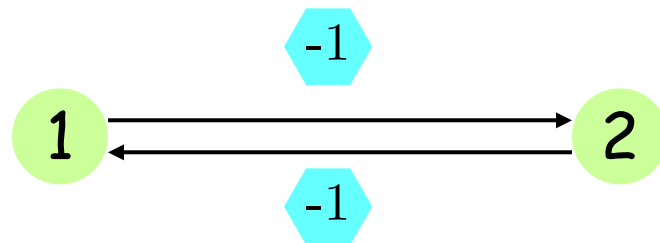
Modes of Operation

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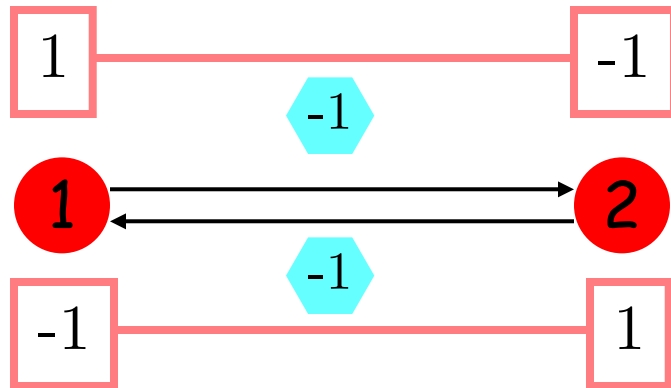
Serial mode: one node at a time (arbitrary order)



Fully-Parallel mode: all nodes at the same time



Stable States



$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} -v_2 \\ -v_1 \end{bmatrix}$$

stable state



$$v_1 = -v_2$$

Q: are those stable states for **serial** or **fully-parallel** modes?

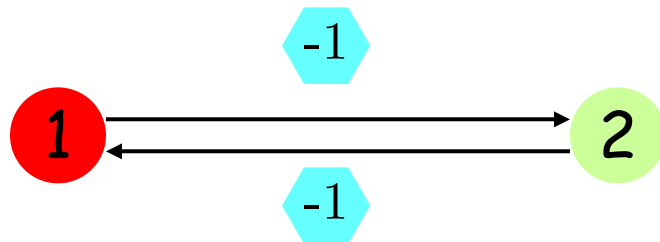
A: a state V is stable iff (it is independent of the mode):

$$V = \text{sgn} \left(W^T \cdot V - T \right)$$

All the nodes must be happy

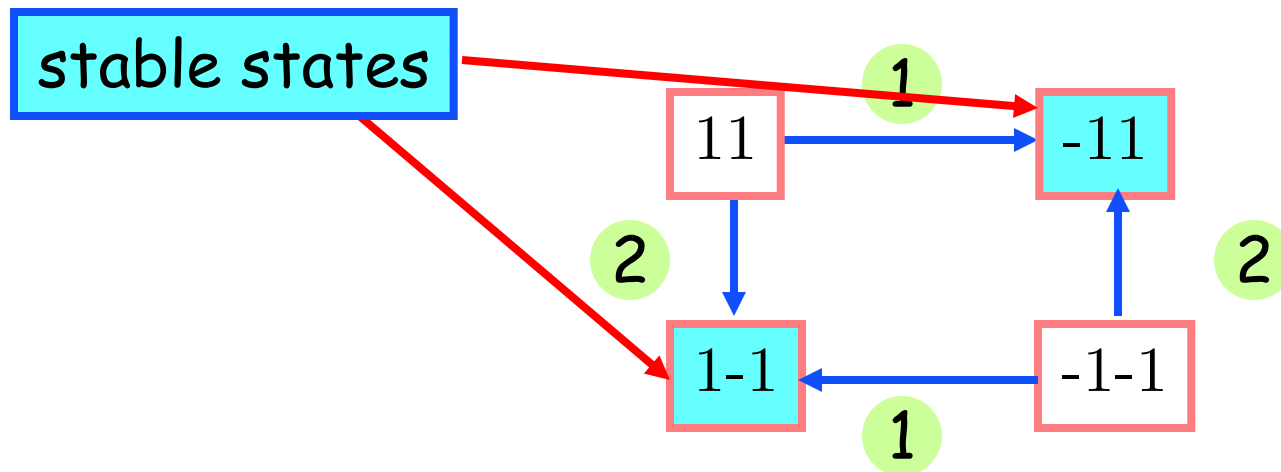
Example 1

Serial Mode - Symmetric Weight Matrix



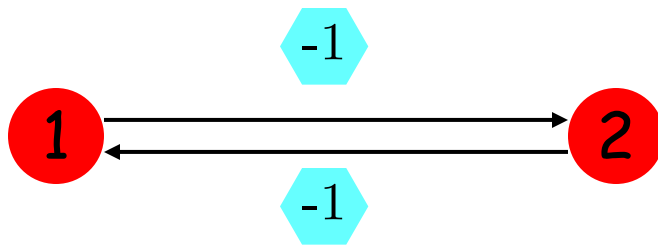
$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The state space:



Example 2

Fully-Parallel Mode - Symmetric Weight Matrix



$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

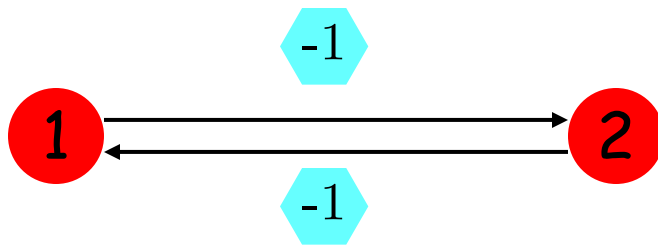
Q: stable states in FP-mode?

stable states are (same as in serial): -11 and 1-1

Q: what happens in the other states?

Example 2

Fully-Parallel Mode - Symmetric Weight Matrix



$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Q: what happens in the other states?

A: start with 11

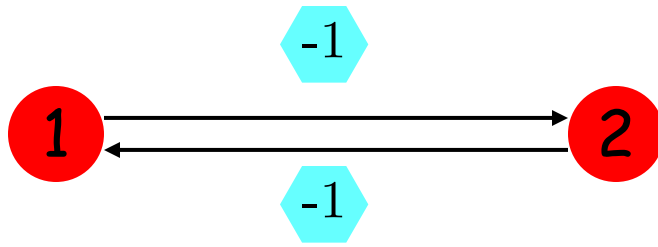
$$\text{sgn} \left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$$

$$\text{sgn} \left(\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

It's a cycle!

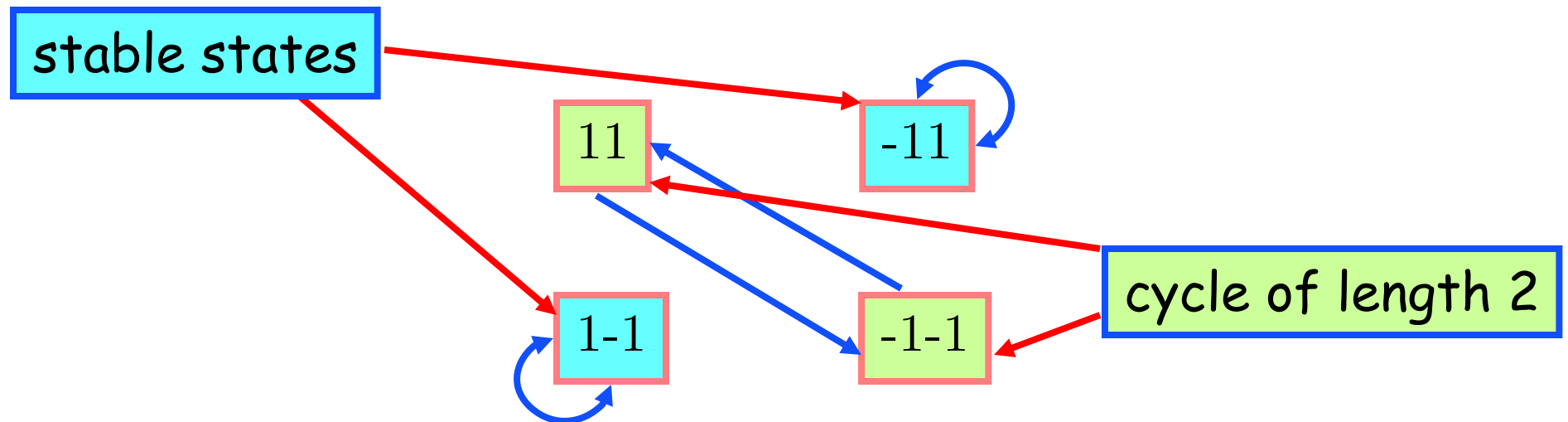
Example 2

Fully-Parallel Mode - Symmetric Weight Matrix



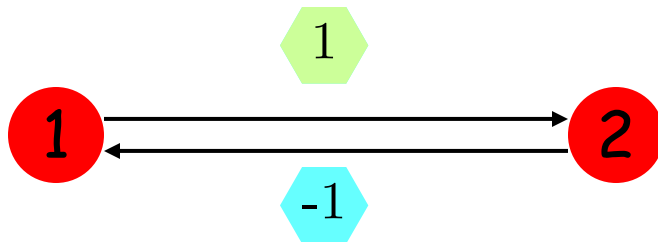
$$W = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The state space:



Example 3

Fully-Parallel Mode Antisymmetric Weight Matrix

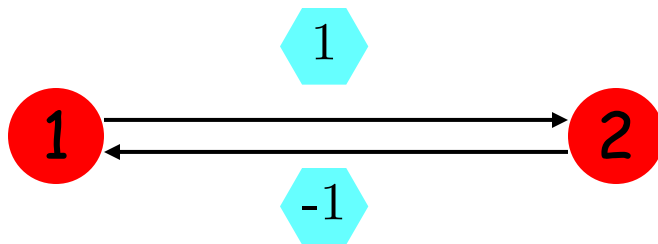


Q: how does the state space look?

$$W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$
$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

Example 3

Fully-Parallel Mode - Antisymmetric Weight Matrix



Q: how does the state space look?

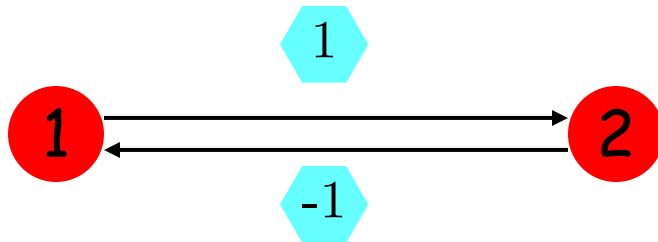
$$T = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

$$\begin{aligned} \text{sgn} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} -1 \\ 1 \end{bmatrix} \\ \text{sgn} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right) &= \begin{bmatrix} -1 \\ -1 \end{bmatrix} \\ \text{sgn} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ -1 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \\ \text{sgn} \left(\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \end{bmatrix} \right) &= \begin{bmatrix} 1 \\ 1 \end{bmatrix} \end{aligned}$$

cycle of length 4

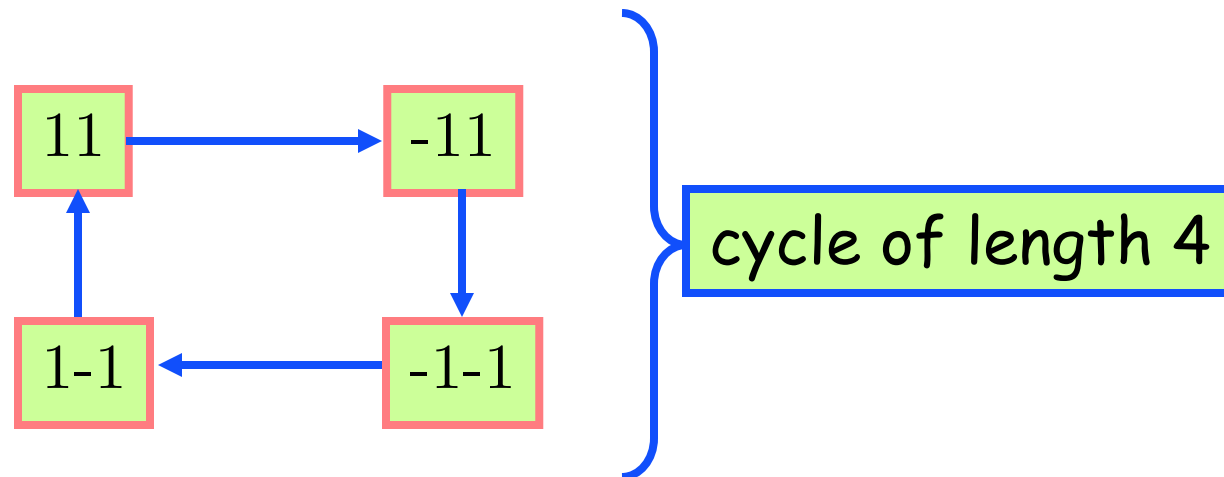
Example 3

Fully-Parallel Mode - Antisymmetric Weight Matrix

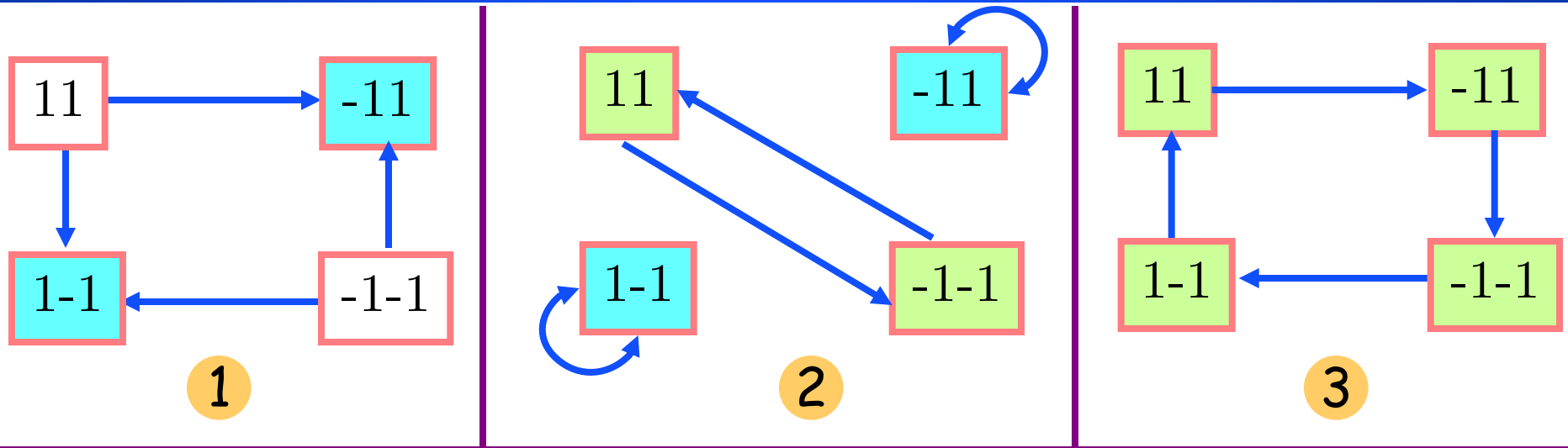


$$T = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \quad W = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

The state space:



The Three Cases



Cycle lengths

Example #

mode \ W	symmetric	antisymmetric
serial	1	?
fully-parallel	1,2	4

The Three Cases

It Is True in General!
It Is True in General!

Cycle lengths

Example #

mode \ W	symmetric	antisymmetric
serial	1	?
fully-parallel	2	3

1 → Hopfield 1982

2 → Goles 1985

3 → Goles 1986

Proof Ideas

Cycle lengths Example #	mode \ W	symmetric	antisymmetric
	serial	1	?
	fully-parallel	2	3

The proofs of these three results use the concept of an **energy function**

For the serial mode:

$$E(t) = V^T(t) \cdot W \cdot V(t) - 2V^T T$$

Show that:

$$E(t+1) - E(t) \geq 0$$

Namely, stable states are **local max** of the **energy** E

Proof Ideas

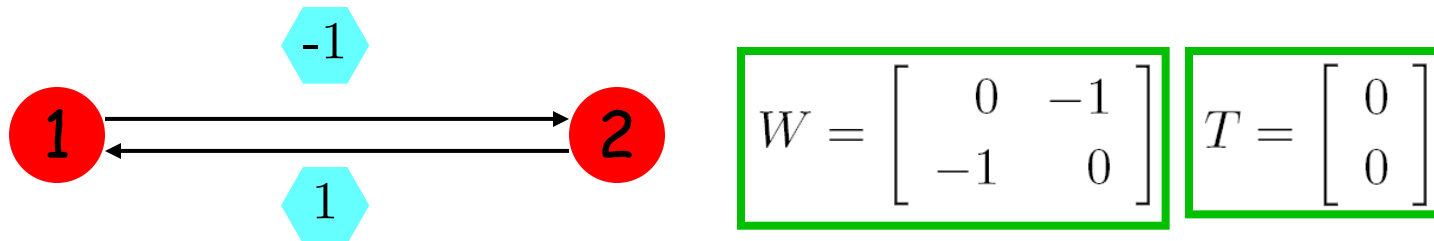
Example 1

The proofs of these three results use the concept of an **energy function**

For the serial mode: $E(t) = V^T(t) \cdot W \cdot V(t) - 2V^T T$

Show that: $E(t+1) - E(t) \geq 0$

Namely, stable states are **local max of the energy** E



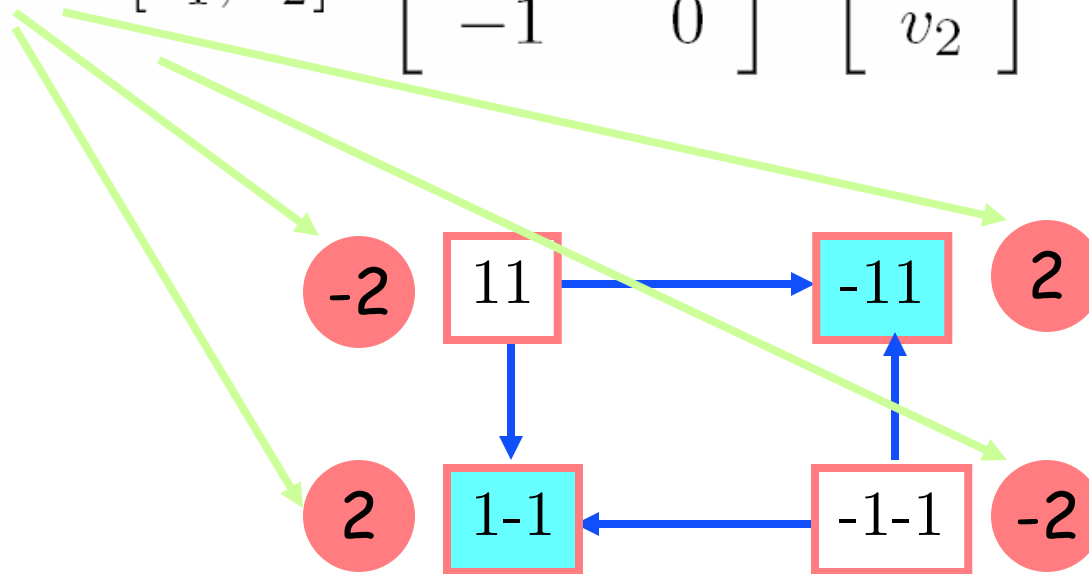
$$E = [v_1, v_2] \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2v_1v_2$$

Proof Ideas

Example 1

Stable states are local max of the energy E

$$E = [v_1, v_2] \cdot \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} \cdot \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = -2v_1v_2$$



Questions on Convergence

Cycle lengths	mode \ W	symmetric	antisymmetric
	serial	1	?
Example #	fully-parallel	2 1,2	3 4

1 → Hopfield 1982

2 → Goles 1985

3 → Goles 1986

Q1: Are the three cases "distinct"?

Q2: Elementary proof? (wo/energy)

Q3: Other "interesting" cases?

NO
YES
e
s