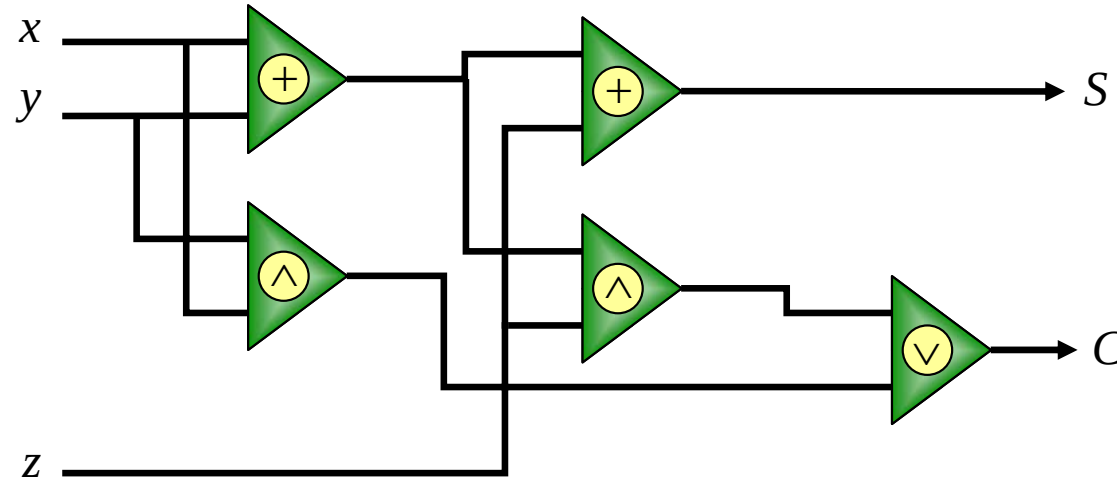
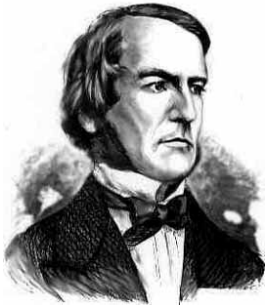


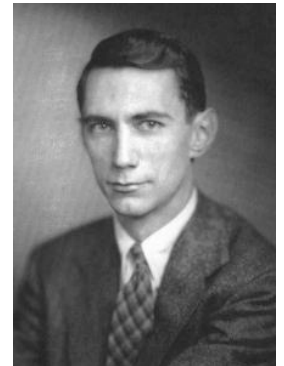
Key to the Progress in Design: Abstractions in Information Systems

Logic to **Boolean Calculus** to Physical Circuits

Boole
1815-1864



Shannon
1916-2001



1847

Connected Logic
with Algebra

Boolean Algebra
Logical Calculation

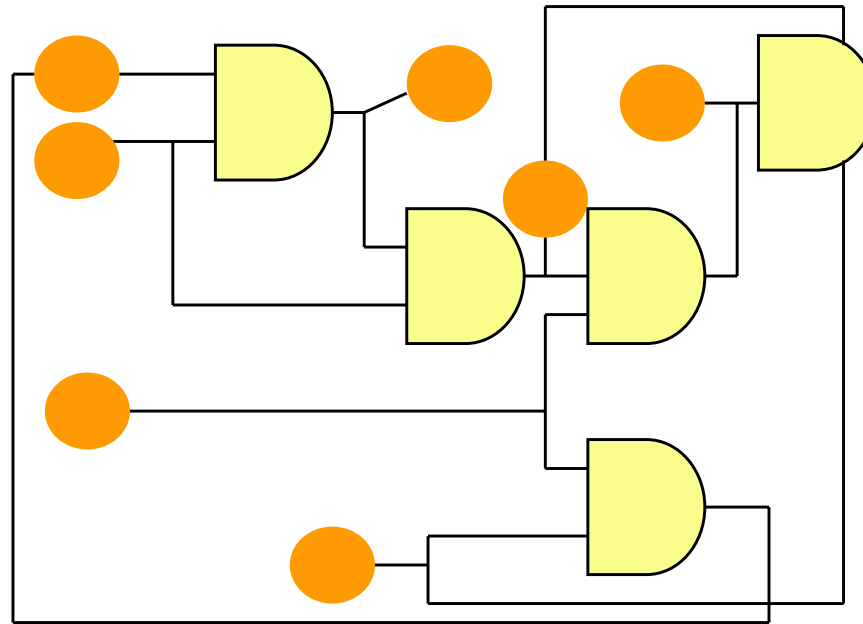
1938

Boolean Algebra to
Electrical Circuits
Logic Design

Key for Analysis

Developing a Calculus for Biology

Abstracting Biological Circuits
to Calculations **to** Reasoning



CNS 188a Overview

- Boolean algebra as an axiomatic system
- Boolean functions and their representations using Boolean formulas and spectral methods
- Implementing Boolean functions with **relay** circuits, circuits of **AON** (AND, OR, NOT) gates and **LT** (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits

Last Time

Q: is there a Boolean algebra with three elements?

Axiomatic definition of Boolean algebras

T0: duality principle

T1: one complement per element

$$L1: a + a = a$$

Examples: 0-1, Boolean integers

The Algebra (Boolean Calculus)

Boole, Jevons, Peirce, Schroder (18xx)
Axiomatic System: Huntington (1904)

Algebraic system: set of elements B ,
two binary operations $+$ and $\cdot$
 B has at least two elements (0 and 1)

If the following axioms are true
then it is a **Boolean Algebra**:

A1. identity

$$a + 0 = a; \quad a \cdot 1 = a$$

A2. complement

$$a + \bar{a} = 1; \quad a \cdot \bar{a} = 0$$

A3. commutative

$$a + b = b + a; \quad a \cdot b = b \cdot a$$

A4. distributive

$$a + b \cdot c = (a + b) \cdot (a + c); \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

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A4. distributive

$$a + (b \cdot c) = (a + b) \cdot (a + c); \quad a \cdot (b + c) = (a \cdot b) + (a \cdot c)$$

Theorems

Theorem 0:

Any identity that is true in a Boolean algebra, is also true if $+$ and $\cdot$ are interchanged, and 0 and 1 are interchanged.

Theorem 1:

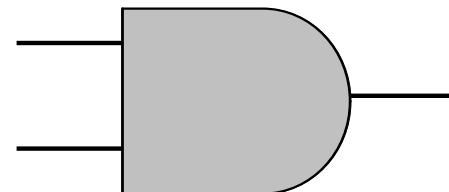
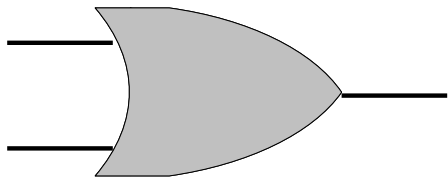
Each element of a Boolean Algebra has exactly one complement.

Two-valued Boolean Algebra

Boolean Algebra: set of elements $B=\{0,1\}$,
two binary operations OR and AND

xy	$OR(x,y)$
00	0
01	1
10	1
11	1

xy	$AND(x,y)$
00	0
01	0
10	0
11	1



Non-Binary Boolean Algebras

Boolean Integers (Bunitskiy 1899)

$$2 \times 3 \times 5 = 30$$

$$2 \times 3 \times 7 = 42$$

Every prime in the prime factorization
is a power of one

The set divisors of a Boolean integer

$\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

Non-Binary Boolean Algebras

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The operations: **lcm** and **gcd**

The special elements: 1 and 30

Q: how many elements in a Bunitskiy Boolean algebra?

Q: Does every Boolean algebra have 2^k elements?

YES

Splitting Fact

Fact (Splitting):

Every Boolean algebra with m (m bigger than 2) elements has a Boolean sub-algebra with $m/2$ elements.

Sub-algebra: subset of the elements in the original algebra that obeys the axioms using the same operations.

Splitting Theorem

The set divisors of a Boolean integer
 $\{1, 2, 3, 5, 6, 10, 15, 30\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 30

All x in B such that

$$a \cdot x = a$$

$$a \prec x$$

What is the 0 element?

It is a

?

$$x + a = x + a \cdot x = x$$

gcd	1	2	3	5	6	10	15	30
1	1	1	1	1	1	1	1	1
2	1	2	1	1	2	2	1	2
3	1	1	3	1	3	1	3	3
5	1	1	1	5	1	5	5	5
6	1	2	3	1	6	2	3	6
10	1	2	1	5	2	10	5	10
15	1	1	3	5	3	5	15	15
30	1	2	3	5	6	10	15	30

Absorption Theorem

Theorem 2: $a + a \cdot b = a$

Absorption Theorem

Theorem 2: $a + a \cdot b = a$

Proof (for 0-1 algebra):

$$\begin{array}{l} 0 + 0 \cdot 0 = 0 \\ 0 + 0 \cdot 1 = 0 \\ 1 + 1 \cdot 0 = 1 \\ 1 + 1 \cdot 1 = 1 \end{array}$$

Absorption Theorem

Theorem 2: $a + a \cdot b = a$
 $a \cdot (a + b) = a$

Proof:

$$a + a \cdot b = a \cdot 1 + a \cdot b \quad \text{A1}$$

$$= a \cdot (1 + b) \quad \text{A4}$$

$$= a \cdot 1 \quad \text{??}$$

$$= a \quad \text{A1}$$

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

Simple Absorption

Lemma 2:

$$1 + a = 1$$
$$0 \cdot a = 0$$

Proof:

$$\begin{aligned} 1 + a &= (1 + a) \cdot 1 && \text{A1} \\ &= (a + 1) \cdot (a + \bar{a}) && \text{A3 A2} \\ &= a + 1 \cdot \bar{a} && \text{A4} \\ &= a + \bar{a} && \text{A3 A1} \\ &= 1 && \text{A2} \end{aligned}$$

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

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Current state

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$

- **L1. Self Absorption:**

$$a + a = a \quad \text{and} \quad a \cdot a = a$$

- **L2: Simple Absorption:**

$$a + 1 = 1 \quad \text{and} \quad a \cdot 0 = 0$$

- **T3. Associativity:**

$$(a + b) + c = a + (b + c)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- **T4. DeMorgan Laws:**

$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

$$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$

- **T0. Duality:**

Correctness is maintained when interchange $+$ and $\cdot$, as well as 0 and 1.

- **T1. Distinct Complement:**

Every element has another element that is its unique complement.

- **T2. Absorption:**

$$a + ab = a \quad \text{and} \quad a \cdot (a + b) = a$$

Homework #1

3. Associativity

In this problem you will prove that the two operations of a Boolean algebra are associative, namely:

$$(a + b) + c = a + (b + c)$$

and

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

Prove the following statements and reach the conclusion.

(a) Prove that:

$$a + a \cdot (b \cdot c) = a + (a \cdot b) \cdot c$$

(b) Prove that:

$$\bar{a} + a \cdot (b \cdot c) = \bar{a} + (a \cdot b) \cdot c$$

(c) Prove that:

$$(a + a \cdot (b \cdot c)) \cdot (\bar{a} + a \cdot (b \cdot c)) = a \cdot (b \cdot c)$$

and

$$(a + (a \cdot b) \cdot c) \cdot (\bar{a} + (a \cdot b) \cdot c) = (a \cdot b) \cdot c$$

(d) Based on (a)-(c) above argue that you proved Associativity.

(e) *Extra credit:* Propose a different (simpler?) proof for Associativity.

Homework #1

4. DeMorgan Laws

(a) Prove that

$$\overline{(a + b)} = \bar{a} \cdot \bar{b}$$

and

$$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$

(b) Find the complement of $a \cdot \bar{b} + \bar{a} \cdot b$.

(c) Prove that $\overline{(a + b + c)} = \bar{a} \cdot \bar{b} \cdot \bar{c}$.

Proof for DeMorgan without
using Associativity?

Examples of Boolean Algebras

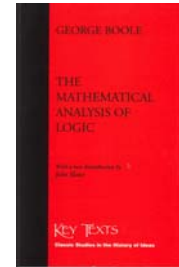
- 0-1 (two valued) Boolean algebra
OR / AND, lecture 1
- Arithmetic Boolean algebras (binary integers)
lcm/gcd, HW #1
- Algebra of subsets
today

Algebra of subsets

A **set** is a collection of **points** in a domain

Universal set: S is the set of all points

For example: S is the set of all books



Subsets:

A = red books

B = blue books

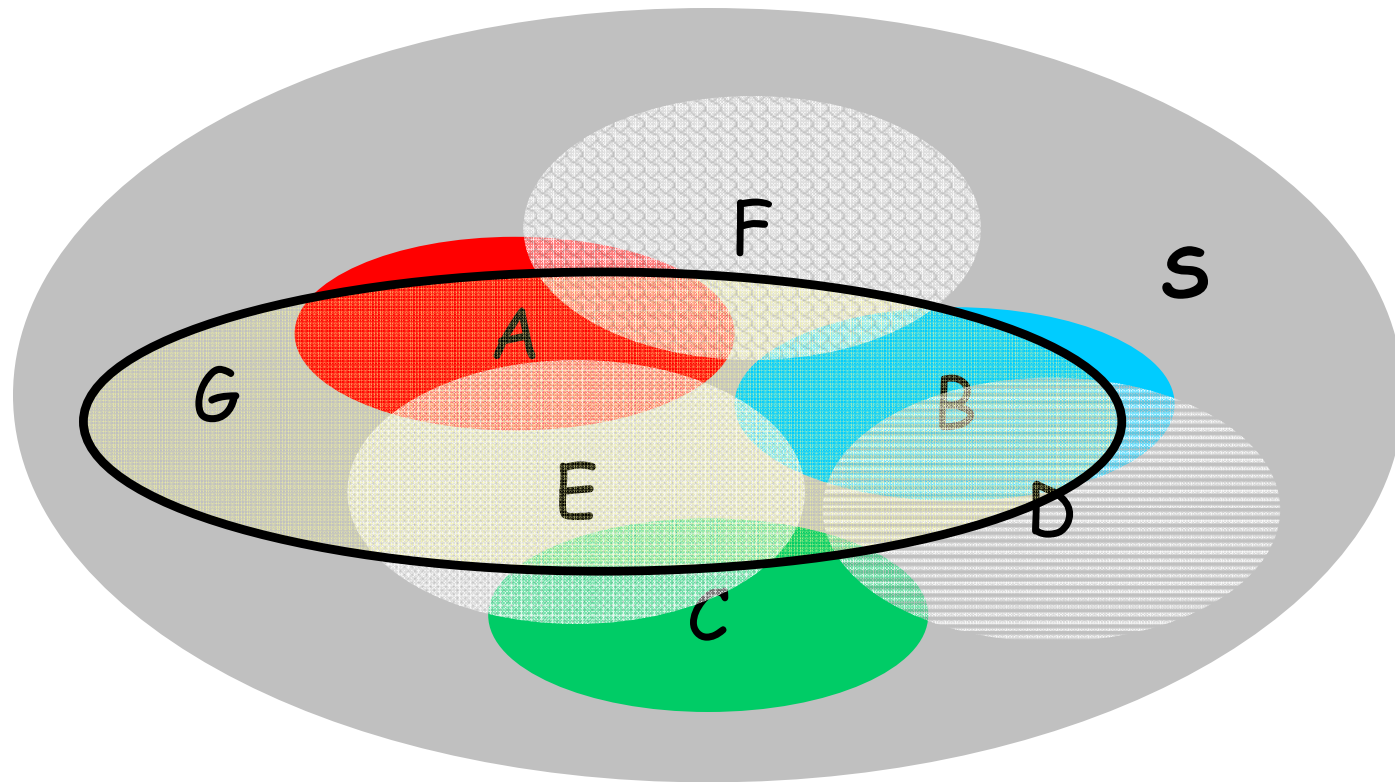
C = green books

D = German books

E = English books

F = French books

G = good books



Venn Diagrams: John Venn 1880

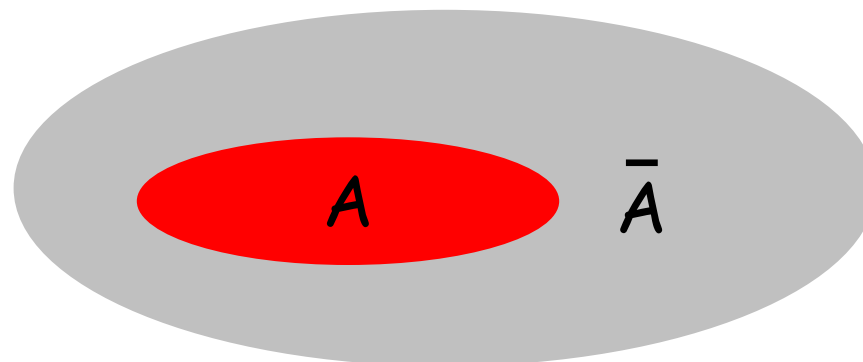
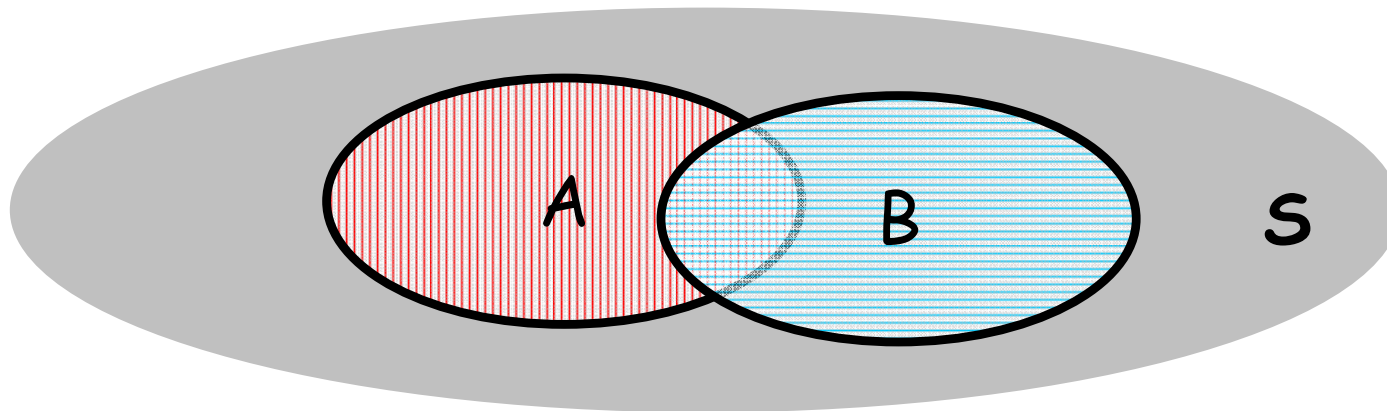
Algebra of subsets: Operations

+ is union of sets: $\cup$

OR

• is intersection of sets: $\cap$

AND



NOT

Algebra of subsets

Elements: all possible subsets of S (there are $2^{|S|}$ subsets)

1 is the set S

0 is the **empty** set

$+$ is **union** of sets

$\cdot$ is **intersection** of sets

Example: $S = \{a, b\}$

Elements: $\emptyset \quad \{a, b\} \quad \{a\} \quad \{b\}$

Algebra of subsets

Elements: $\emptyset$ $\{a, b\}$ $\{a\}$ $\{b\}$

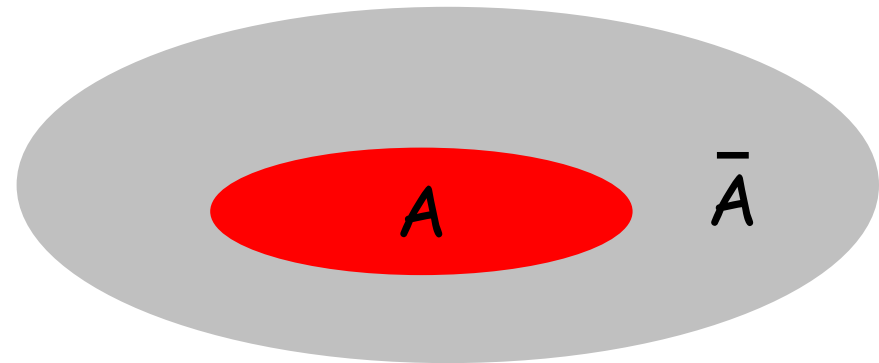
Complement:

$$\overline{\{a, b\}} = \emptyset$$

$$\overline{\emptyset} = \{a, b\}$$

$$\overline{\{a\}} = \{b\}$$

$$\overline{\{b\}} = \{a\}$$



Algebra of subsets

$\cup$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$	$\{b\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

Algebra of subsets

$\cap$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a, b\}$	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$

Is the Algebra of Subsets Boolean?

- **A1. Identities:**

$$a + 0 = a \quad \text{and} \quad a \cdot 1 = a$$

- **A2. Complements:**

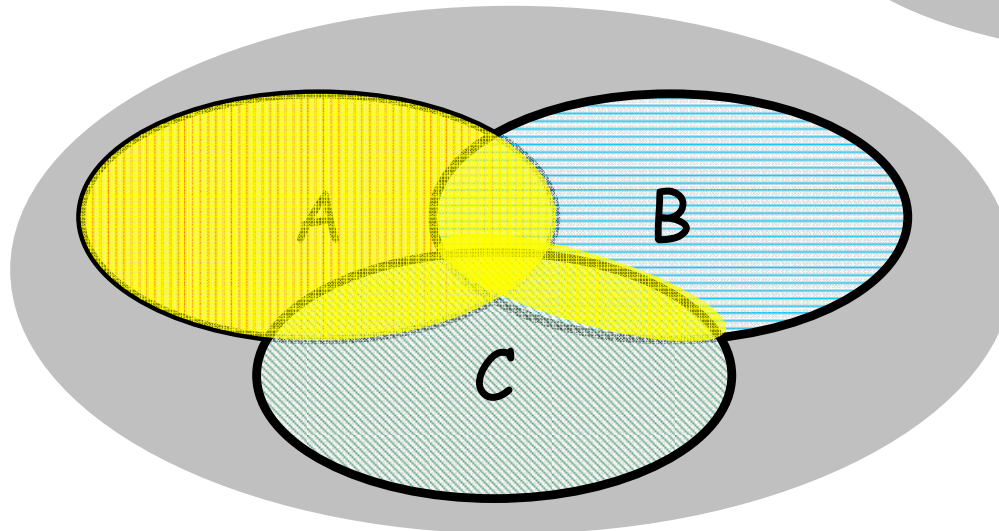
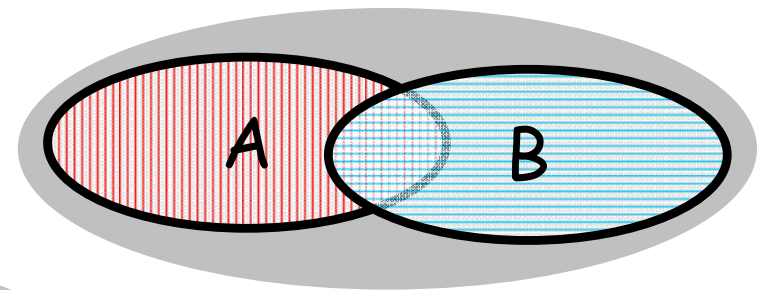
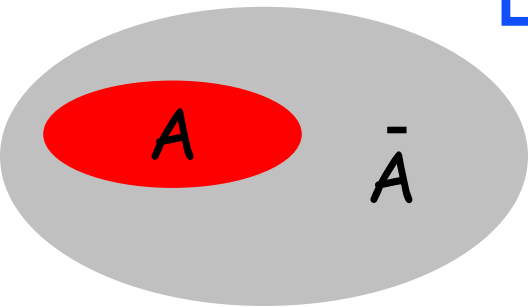
$$a + \bar{a} = 1 \quad \text{and} \quad a \cdot \bar{a} = 0$$

- **A3. Commutativity:**

$$a + b = b + a \quad \text{and} \quad a \cdot b = b \cdot a$$

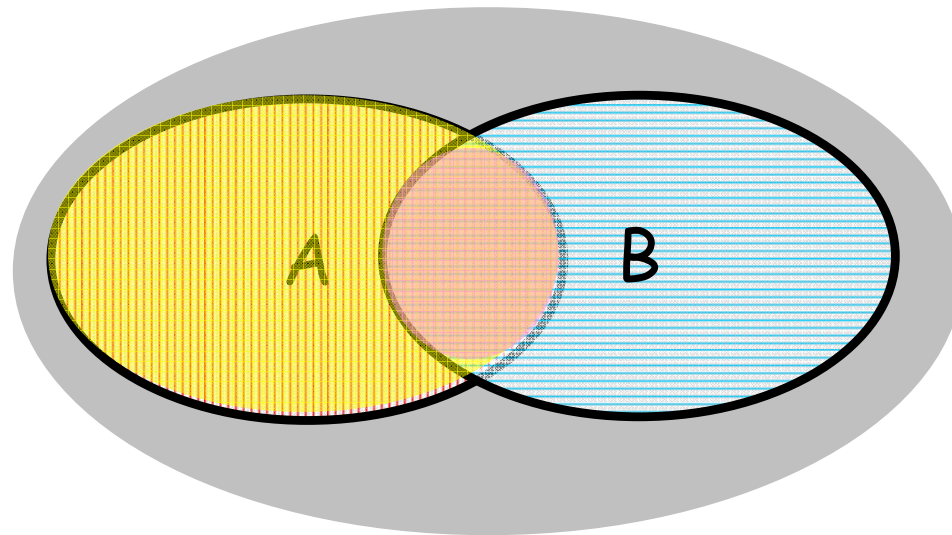
- **A4. Distributivity:**

$$a + b \cdot c = (a + b) \cdot (a + c) \quad \text{and} \quad a \cdot (b + c) = a \cdot b + a \cdot c$$



Absorption Theorem

Theorem 2: $a + a \cdot b = a$



More Theorems

- **T3. Associativity:**

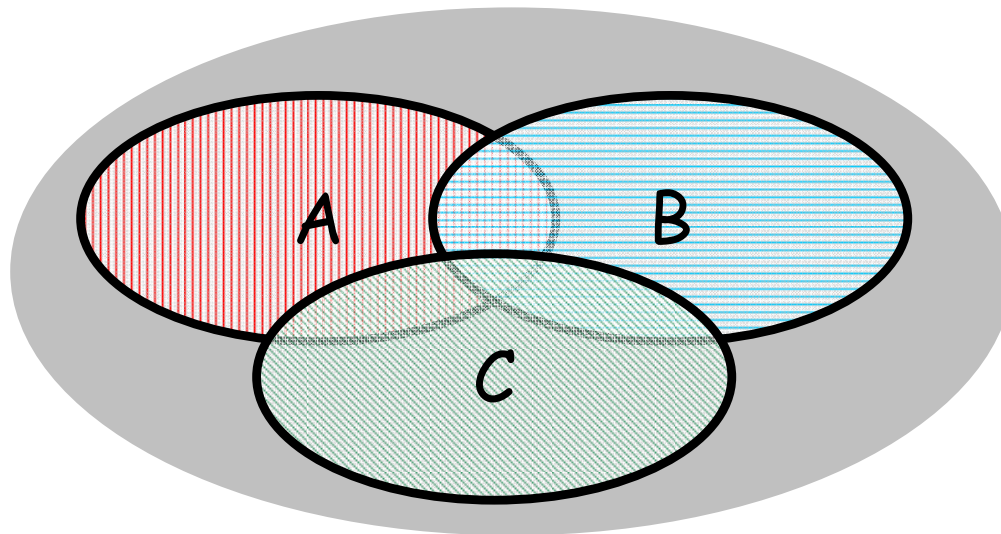
$$(a + b) + c = a + (b + c)$$

$$(a \cdot b) \cdot c = a \cdot (b \cdot c)$$

- **T4. DeMorgan Laws:**

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$$\overline{(a \cdot b)} = \bar{a} + \bar{b}$$



More Theorems

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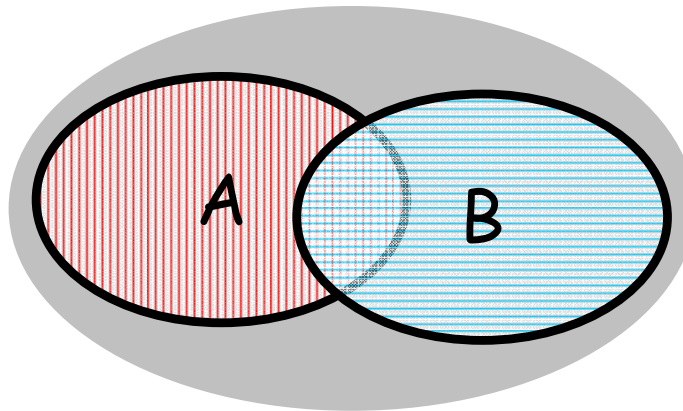
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The Algebra of Google

S is the set of all the words in English

A web page is an element in the algebra

The Google logo, consisting of the word "Google" in its characteristic multi-colored font.The Google logo, consisting of the word "Google" in its characteristic multi-colored font.The Google logo, consisting of the word "Google" in its characteristic multi-colored font.

The Second Edition of the *Oxford English Dictionary* contains full entries for **171,476 words** in current use, and **47,156 obsolete words**.

Over half of these words are nouns, about a quarter adjectives, and about a seventh verbs

Algebra of Google

S is the set of all the words in English

A web page is an element in the algebra

Google

Google

Google

A search is finding all the subsets (web pages) that contain the subset defined by the query

Algebra of Google

S is the set of all the words in English

A web page is an element in the algebra

A query is an element in the algebra



AND → with **all** of the words

Find results with the **exact phrase**

OR → with **at least one** of the words

without the words

↑
NOT

10 results ▼

Google Search

An Amazing Theorem

Representation Theorem (Stone 1936):

Every *finite* Boolean algebra is isomorphic to a Boolean algebra of subsets of some *finite* set S .

Algebra 1

elements

Algebra 2

elements



operations/elements



operations/elements

Arithmetic Boolean Algebras

Isomorphic to Algebra of Subsets

The set of divisors of a Boolean integer
 $\{1, 2, 3, 6\}$

The operations: **lcm** and **gcd**

The special elements: 1 and 6

Isomorphic to: $\emptyset \quad \{a, b\} \quad \{a\} \quad \{b\}$

$$(x_1, x_2) \equiv 2^{x_1} \times 3^{x_2}$$

Arithmetic / Subsets

$\bigcup$ lcm	$\emptyset$ 1	$\{a\}$ 2	$\{b\}$ 3	$\{a, b\}$ 6
$\emptyset$ 1	$\emptyset$ 1	$\{a\}$ 2	$\{b\}$ 3	$\{a, b\}$ 6
$\{a\}$ 2	$\{a\}$ 2	$\{a\}$ 2	$\{a, b\}$ 6	$\{a, b\}$ 6
$\{b\}$ 3	$\{b\}$ 3	$\{a, b\}$ 6	$\{b\}$ 3	$\{a, b\}$ 6
$\{a, b\}$ 6	$\{a, b\}$ 6	$\{a, b\}$ 6	$\{a, b\}$ 6	$\{a, b\}$ 6

Arithmetic / Subsets

$\bigcap$ gcd	$\emptyset$ 1	$\{a\}$ 2	$\{b\}$ 3	$\{a, b\}$ 6
$\emptyset$ 1	$\emptyset$ 1	$\emptyset$ 1	$\emptyset$ 1	$\emptyset$ 1
$\{a\}$ 2	$\emptyset$ 1	$\{a\}$ 2	$\emptyset$ 1	$\{a\}$ 2
$\{b\}$ 3	$\emptyset$ 1	$\emptyset$ 1	$\{b\}$ 3	$\{b\}$ 3
$\{a, b\}$ 6	$\emptyset$ 1	$\{a\}$ 2	$\{b\}$ 3	$\{a, b\}$ 6

An Amazing Theorem

Representation Theorem (Stone 1936):

Every *finite* Boolean algebra is isomorphic to a Boolean algebra of subsets of some *finite* set S .

Two Boolean Algebras with m elements are isomorphic

Every Boolean algebra has 2^k elements

Intuition beyond the axioms:

We can 'naturally' reason about results in Boolean algebra

Algebra of Subsets and Vectors

Elements: all possible subsets of S (there are $2^{|S|}$ subsets)

As subset can be represented by a binary vector of length $|S|$
where a 0 represents that a point is not in the set
a 1 represents that a point is in the set

Example: $S = \{a, b\}$

Elements: $\emptyset \quad \{a, b\} \quad \{a\} \quad \{b\}$

Elements are: (00), (11), (10), (01)

Algebra of Subsets and Vectors

Elements: all possible subsets of S (there are $2^{|S|}$ subsets)

As subset can be represented by a binary vector of length $|S|$
where a 0 represents that a point is not in the set
a 1 represents that a point is in the set

- + is union of sets: $\cup$
- is intersection of sets: $\cap$

- Union corresponds to bitwise OR on the vectors
- Intersection corresponds to bitwise AND on the vectors

Example: Bitwise Binary OR

Elements are: (00), (11), (10), (01)

$\cup$	$\emptyset$ 00	$\{a\}$ 10	$\{b\}$ 01	$\{a, b\}$ 11
$\emptyset$ 00	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$
$\{a\}$ 10	$\{a\}$	$\{a\}$	$\{a, b\}$	$\{a, b\}$
$\{b\}$ 01	$\{b\}$	$\{a, b\}$	$\{b\}$	$\{a, b\}$
$\{a, b\}$ 11	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$	$\{a, b\}$

Example: Bitwise Binary OR

Elements are: (00), (11), (10), (01)

	$\emptyset$ 00	$\{a\}$ 10	$\{b\}$ 01	$\{a, b\}$ 11
$\emptyset$ 00	$\emptyset$	$\emptyset$	$\emptyset$	$\emptyset$
$\{a\}$ 10	$\emptyset$	$\{a\}$	$\emptyset$	$\{a\}$
$\{b\}$ 01	$\emptyset$	$\emptyset$	$\{b\}$	$\{b\}$
$\{a, b\}$ 11	$\emptyset$	$\{a\}$	$\{b\}$	$\{a, b\}$

The 0-1 Theorem

0-1 Theorem:

An identity is true for any finite Boolean algebra if and only if it is true for a two valued (0-1) Boolean algebra.

Proof: The easy direction

Assume an identity true for any finite Boolean algebra

 True for 0-1 Boolean algebra

The 0-1 Theorem

0-1 Theorem:

An identity is true for any finite Boolean algebra if and only if it is true for a two valued (0-1) Boolean algebra.

Proof:

The non-obvious direction

Assume an identity true for a 0-1 Boolean algebra

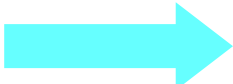
Need to prove true for any finite Boolean algebra

The key: Stone's representation theorem

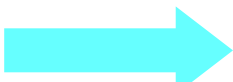
The 0-1 Theorem

Proof (cont.): Assume an identity true for 0-1 Boolean algebra
Need to prove true for any finite Boolean algebra
The key: Stone's representation theorem

Assume that there exists an identity that is **true** for 0-1 Boolean algebra but is **not true** for some other Boolean algebra.



By Stone: The identity is not true for some assignment of **binary vectors** with bitwise OR / AND



There exists a **0-1 assignment** to the identity that violates the equality, **CONTRADICTION!!**

"The further back you look, the further forward you can see"

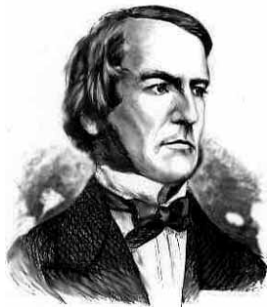
Winston Churchill

Gottfried Leibniz
1646-1716



- Logic and binary system
- Calculus

George Boole
1815-1864



Connected Logic
with Algebra
Boolean Algebra
Logical Calculation

Edward Huntington
1874-1952



Axiomatic systems
for Boolean Algebra

Marshall Stone
1903-1989



Boolean
representation
theorem

Marshal Stone

Marshal Stone
1903-1989



Boolean
representation
theorem

**Born: 8 April 1903 in New York, USA
Died: 9 Jan 1989 in Madras, India**

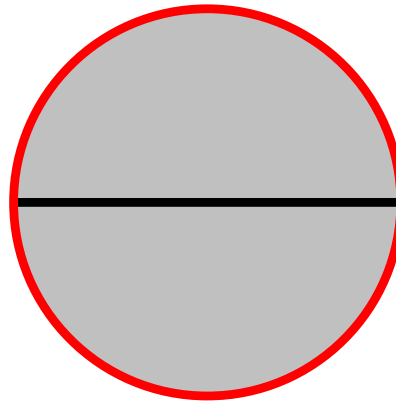
Leibniz

Gottfried Leibniz
1646-1716



- Leibniz's first great mathematical success was his creation of a calculating machine in the early 1670's.
- It was apparently the first machine capable of multiplication, division, and root extractions.
- It was impressive enough to get Leibniz elected to the Royal Society of London in 1673.

More History (Leibniz 1670's)



3.14159265...

π = The circumference of a circle with diameter 1

Leibniz Showed how to calculate π to an arbitrary precision
Gregory (1638- 1675)

$$\frac{\pi}{4} = 1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \frac{1}{9} - \frac{1}{11} + \dots$$