

# CNS 188a Overview

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- Boolean algebra as an axiomatic system
- Boolean functions and their representations using Boolean formulas and spectral methods
- Implementing Boolean functions with relay circuits, circuits of AON (AND, OR, NOT) gates and LT (Linear Threshold) gates
- Analyzing the complexity (size and depth) of circuits
- Relations (as opposed to functions) and their implementation in circuits
- Feedback and convergence in LT circuits

# Spectral Analysis of Boolean Functions

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Q: why bother?

- Upper and lower bounds on the size of LT circuits
- Synthesis techniques for LT circuits
- Linear vs. Polynomial threshold functions
- Best known learning algorithms for Boolean functions

# Representations of Boolean Functions

We will represent the 0 element by 1  
and the 1 element by -1

$$a \mapsto (-1)^a$$

$$f : \{1, -1\}^n \mapsto \{1, -1\}$$

The AND function of two variables:

**DNF:**  $f(x_1, x_2) = x_1 \cdot x_2$

**Truth table:**

| $x_1x_2$ | $f(x_1, x_2)$ |
|----------|---------------|
| 11       | 1             |
| 1-1      | 1             |
| -11      | 1             |
| -1-1     | -1            |

**LT function:**

$$f(x_1, x_2) = \text{sgn}(1 + x_1 + x_2)$$

| $x_1x_2$ | $\downarrow$ |
|----------|--------------|
| 11       | 3            |
| 1-1      | 1            |
| -11      | 1            |
| -1-1     | -1           |

# The star of the Show: Polynomial Representation

Use normal  
addition  
subtraction  
multiplication  
division

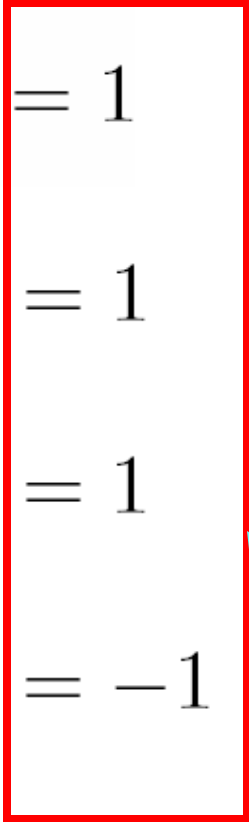
$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

$$f(1, 1) = \frac{1}{2}(1 + 1 + 1 - 1) = 1$$

$$f(1, -1) = \frac{1}{2}(1 + 1 - 1 + 1) = 1$$

$$f(-1, 1) = \frac{1}{2}(1 - 1 + 1 + 1) = 1$$

$$f(-1, -1) = \frac{1}{2}(1 - 1 - 1 - 1) = -1$$



AND

# Polynomial Representation

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

$$\frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \quad -\frac{1}{2}$$

**The spectrum**

is the set of the coefficients of the polynomial representation

**Important fact:** it works for arbitrary Boolean functions

# Polynomial Representation

Works for OR

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

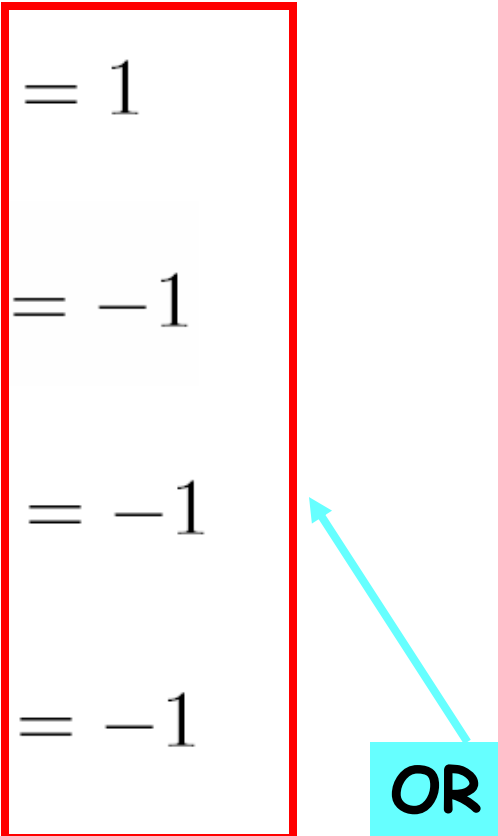
$$f(1, 1) = \frac{1}{2}(-1 + 1 + 1 + 1) = 1$$

$$f(1, -1) = \frac{1}{2}(-1 + 1 - 1 - 1) = -1$$

$$f(-1, 1) = \frac{1}{2}(-1 - 1 + 1 - 1) = -1$$

$$f(-1, -1) = \frac{1}{2}(-1 - 1 - 1 + 1) = -1$$

OR



# Polynomial Representation

Works for AND, OR and ...

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AND

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

OR

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

NOT

$$f(x_1) = -x_1$$

XOR

$$f(x_1, x_2, \dots, x_n) = x_1x_2 \cdots x_n$$

# Polynomial Representation

## Works for any Boolean Function

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Q: It works for AND, OR and NOT  
Does it work for any Boolean function?  
Is it unique?

AND

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

OR

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

NOT

$$f(x_1) = -x_1$$

# Spectral Analysis of Boolean Functions

## Notation

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An efficient way to represent a multilinear monomial (term)

$$x^2 = 1 \text{ for } x \in \{-1, 1\}$$

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n}$$

$$\alpha \in \{0, 1\}^n$$

For example:

$$\alpha = 11 \longrightarrow X^\alpha = x_1 x_2$$

$$\alpha = 00 \longrightarrow X^\alpha = 1$$

# Spectral Analysis of Boolean Functions Notation

---

An efficient way to represent a multilinear monomial (term)

$$X^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \dots x_n^{\alpha_n} \quad \alpha \in \{0, 1\}^n$$

The AND example:

$$f(x_1, x_2) = \frac{1}{2}(X^{00} + X^{10} + X^{01} - X^{11})$$

$\downarrow$

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

The coefficients:

$$a_{00} \quad a_{10} \quad a_{01} \quad a_{11}$$

# Polynomial Representation Theorem

## Representation Theorem:

(Fourier 18xx, Plancherel 19xx, Muller 195x, Ninomiya 195x)

(i) Every Boolean function has a unique representation as a multilinear polynomial of the form:

$$X = (x_1, x_2, \dots, x_n)$$

$$f : \{1, -1\}^n \mapsto \{1, -1\}$$

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

spectrum of the function

# Polynomial Representation Theorem

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## Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Sylvester type  
Hadamard matrix

The coefficients listed in lexicographic order

The function listed in lexicographic order

# Sylvester Type Hadamard Matrix

J. J. Sylvester, *Thoughts on Inverse Orthogonal Matrices, Simultaneous Sign Successions, and Tessellated Pavements in Two or More Colors*, Phil, Mag., 34, pp. 461–475, 1867.

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Recursive definition of  $H$ :

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

# Sylvester Type Hadamard Matrix

## Example

$$H_4 = \left[ \begin{array}{cc|cc} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ \hline 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{array} \right]$$

Recursive definition:

$$H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

# Polynomial Representation Theorem

## Example

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} f_n$$

The AND function:

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0.5 \\ 0.5 \\ 0.5 \\ -0.5 \end{bmatrix}$$

$$f(x_1, x_2) = \frac{1}{2}(1 + x_1 + x_2 - x_1x_2)$$

# Polynomial Representation Theorem

## Example

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} f_n$$

The OR function:

$$\frac{1}{4} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ -1 \\ -1 \\ -1 \end{bmatrix} = \begin{bmatrix} -0.5 \\ 0.5 \\ 0.5 \\ 0.5 \end{bmatrix}$$

$$f(x_1, x_2) = \frac{1}{2}(-1 + x_1 + x_2 + x_1x_2)$$

# Polynomial Representation Theorem

## Proof

Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Proof: ??

**Idea:** we will compute the spectrum and show that the solution is **unique**

How to compute the spectrum?

**Idea:** solve a system of equations...

# Polynomial Representation Theorem

## Proof

### Representation Theorem:

(ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

**Proof:** **Idea:** we will compute the spectrum by solving a system of equations

$n=1$   $f(x_1) = a_0 + a_1 x_1$   $\longrightarrow$   $f(1) = a_0 + a_1$  **linear**

$\searrow$   $f(-1) = a_0 - a_1$



$$a_0 = \frac{f(1) + f(-1)}{2}$$

$$a_1 = \frac{f(1) - f(-1)}{2}$$

# Polynomial Representation Theorem

## Proof

### Representation Theorem:

(ii) The spectrum can be computed as follows:

**Proof:**

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Does it work for  $n=1$  ?

$$a_0 = \frac{f(1) + f(-1)}{2}$$

$$a_1 = \frac{f(1) - f(-1)}{2}$$

YES!

$$\begin{bmatrix} a_0 \\ a_1 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f(1) \\ f(-1) \end{bmatrix}$$

**Plan:**  
Use induction

# Polynomial Representation Theorem

## Proof

### Representation Theorem:

(ii) The spectrum can be computed as follows:

**Proof:**

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Does it work for  $n=2$  ?

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

$$f(1, 1) = a_{00} + a_{10} + a_{01} + a_{11}$$

$$f(-1, 1) = a_{00} - a_{10} + a_{01} - a_{11}$$

$$f(1, -1) = a_{00} + a_{10} - a_{01} - a_{11}$$

$$f(-1, -1) = a_{00} - a_{10} - a_{01} + a_{11}$$

???

# Polynomial Representation Theorem

## Proof

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

$$f(1, 1) = a_{00} + a_{10} + a_{01} + a_{11}$$

$$f(-1, 1) = a_{00} - a_{10} + a_{01} - a_{11}$$

$$f(1, -1) = a_{00} + a_{10} - a_{01} - a_{11}$$

$$f(-1, -1) = a_{00} - a_{10} - a_{01} + a_{11}$$

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$\begin{bmatrix} f(1, 1) \\ f(-1, 1) \\ f(1, -1) \\ f(-1, -1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}$$

# Polynomial Representation Theorem

## Proof

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

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# Polynomial Representation Theorem

## Proof

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

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$$\begin{bmatrix} f(1, 1) \\ f(-1, 1) \\ f(1, -1) \\ f(-1, -1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}$$

# Polynomial Representation Theorem

## Proof

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

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# Polynomial Representation Theorem

## Proof

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

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$$f(1, -1) = a_{00} + a_{10} - a_{01} - a_{11}$$

$$f(-1, -1) = a_{00} - a_{10} - a_{01} + a_{11}$$

It seems that?!??

$$\underline{f_n} = H_{2^n} A_n(f)$$

$$\begin{bmatrix} f(1, 1) \\ f(-1, 1) \\ f(1, -1) \\ \underline{f(-1, -1)} \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ \underline{1 & -1 & -1 & 1} \end{bmatrix} \begin{bmatrix} \underline{a_{00}} \\ a_{10} \\ a_{01} \\ \underline{a_{11}} \end{bmatrix}$$

# Polynomial Representation Theorem

## Proof

$$\begin{bmatrix} f(1, 1) \\ f(-1, 1) \\ f(1, -1) \\ f(-1, -1) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a_{00} \\ a_{10} \\ a_{01} \\ a_{11} \end{bmatrix}$$

It seems that?!!??

$$\underline{f_n} = H_{2^n} A_n(f)$$

However, we need to prove that:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

We'll first prove that:

$$\underline{f_n} = H_{2^n} A_n(f)$$

The proof is by **induction**

We proved the base case:  $n=1$  as well as  $n=2$

# Polynomial Representation Theorem

## Proof

We'll first prove that:

$$\underline{f_n} = H_{2^n} A_n(f)$$

The proof is by **induction**

We proved the base case:  $n=1$  as well as  $n=2$

Assume that:

$$\underline{f_n} = H_{2^n} A_n(f)$$

Need to prove that:

$$\underline{f_{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix} A_{n+1}(f) = H_{2^{n+1}} A_{n+1}(f)$$

# Polynomial Representation Theorem

## Proof

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$$f_1$$

$$x_2 f_1$$

$$f(x_1, x_2) = a_{00} + a_{10}x_1 + a_{01}x_2 + a_{11}x_1x_2$$

$$f(1, 1) = a_{00} + a_{10} + a_{01} + a_{11}$$

$$f(-1, 1) = a_{00} - a_{10} + a_{01} - a_{11}$$

$$f(1, -1) = a_{00} + a_{10} - a_{01} - a_{11}$$

$$f(-1, -1) = a_{00} - a_{10} - a_{01} + a_{11}$$

$$x_2 = 1$$

$$x_2 = -1$$

the only quadrant that changes sign

# Polynomial Representation Theorem

## Proof

The proof is by **induction**

We proved the base case:  $n=1$  as well as  $n=2$

Assume that:

$$\underline{f}_n = H_{2^n} A_n(f)$$

Need to prove that:

$$x_{n+1} = 1$$

$$\underline{f}_{n+1} =$$

$$x_{n+1} = -1$$

$$\begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix} A_{n+1}(f)$$

$$\underline{f}_n = H_{2^n} A_n(f)$$

Q

Part of the polynomial  
without  $x_{n+1}$

Part of the polynomial  
with  $x_{n+1}$

However we need to prove that:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f}_n$$

# Polynomial Representation Theorem

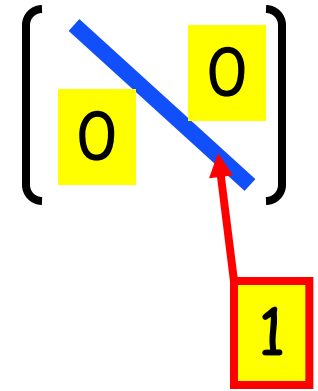
## Proof

**Lemma:**

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

Where  $I_{2^n}$  is the identity matrix of dimension  $2^n$

**Proof:**



BY INDUCTION

# Polynomial Representation Theorem

## Proof

**Lemma:**

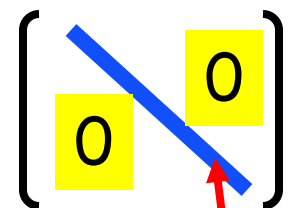
$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

Where  $I_{2^n}$  is the identity matrix of dimension  $2^n$

**Proof:**

$$n = 1$$

$$H_2 \cdot H_2 = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} = 2I_2$$



$$\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Assume true for  $n-1$  and prove for  $n$ :

$$H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^{n-1} I_{2^{n-1}}$$

$$H_{2^n} \cdot H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \cdot \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}$$

# Polynomial Representation Theorem

## Proof

**Lemma:**

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

**Proof:**

Assume true for  $n-1$  and prove for  $n$ :

$$H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^{n-1} I_{2^{n-1}}$$

$$H_{2^n} \cdot H_{2^n} = \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \cdot \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix}$$

$$2H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^n I_{2^{n-1}}$$

# Polynomial Representation Theorem

## Proof

**Lemma:**

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

**Proof:**

Assume true for  $n-1$  and prove for  $n$ :

$$H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^{n-1} I_{2^{n-1}}$$

$$H_{2^n} \cdot H_{2^n} = \begin{bmatrix} \boxed{H_{2^{n-1}} & H_{2^{n-1}}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \cdot \begin{bmatrix} H_{2^{n-1}} & \boxed{H_{2^{n-1}}} \\ H_{2^{n-1}} & -\boxed{H_{2^{n-1}}} \end{bmatrix}$$

$$\boxed{H_{2^{n-1}} \cdot H_{2^{n-1}} - H_{2^{n-1}} \cdot H_{2^{n-1}} = 0}$$

# Polynomial Representation Theorem

## Proof

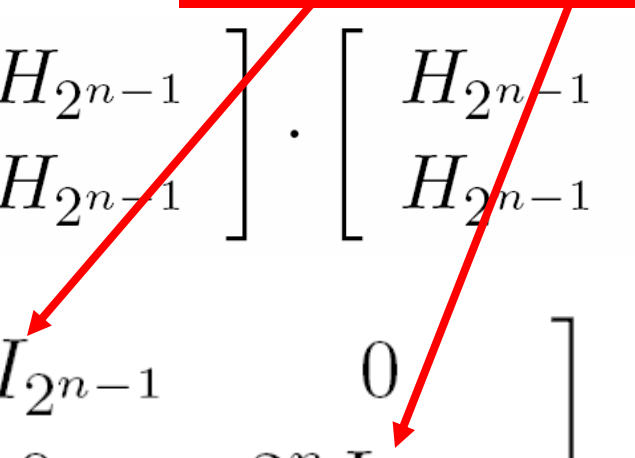
**Lemma:**

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

**Proof:**

Assume true for  $n-1$  and prove for  $n$ :

$$H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^{n-1} I_{2^{n-1}}$$

$$\begin{aligned} H_{2^n} \cdot H_{2^n} &= \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \cdot \begin{bmatrix} H_{2^{n-1}} & H_{2^{n-1}} \\ H_{2^{n-1}} & -H_{2^{n-1}} \end{bmatrix} \\ &= \begin{bmatrix} 2^n I_{2^{n-1}} & 0 \\ 0 & 2^n I_{2^{n-1}} \end{bmatrix} \end{aligned}$$


# Polynomial Representation Theorem

## Proof

**Lemma:**

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

**Proof:**

Assume true for  $n-1$  and prove for  $n$ :

$$H_{2^{n-1}} \cdot H_{2^{n-1}} = 2^{n-1} I_{2^{n-1}}$$

$$H_{2^n} \cdot H_{2^n} = \begin{bmatrix} 2^n I_{2^{n-1}} & 0 \\ 0 & 2^n I_{2^{n-1}} \end{bmatrix}$$

$$= 2^n I_{2^n}$$

# Polynomial Representation Theorem

## Putting it all together

We proved that:

$$\underline{f_n} = H_{2^n} A_n(f)$$

And also (Lemma):

$$H_{2^n} \cdot H_{2^n} = 2^n I_{2^n}$$

Need to proved that:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

Multiply both  
Sides by:

$$H_{2^n} \cdot$$

$$\underline{f_n} = H_{2^n} A_n(f)$$

$$H_{2^n} \underline{f_n} = H_{2^n} \cdot H_{2^n} A_n(f)$$

By the lemma:

$$H_{2^n} \underline{f_n} = 2^n I_{2^n} A_n(f)$$

Q

# Polynomial Representation Theorem

## Representation Theorem:

(Fourier 18xx, Plancherel 19xx, Muller 195x, Ninomiya 195x)

- (i) Every Boolean function has a unique representation as a multilinear polynomial of the form:

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

- (ii) The spectrum can be computed as follows:

$$A_n(f) = \frac{1}{2^n} H_{2^n} \underline{f_n}$$

# Polynomial Representation

## Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

**P1. Orthogonality:**

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

**P2. Zero-Value:**

$$\sum_{\alpha \in \{0,1\}^n} a_{\alpha} = f(1, 1, \dots, 1) \in \{1, -1\}$$

**P3. Moments:**

$$\sum_{X \in \{1,-1\}^n} f(X) X^{\alpha} = 2^n a_{\alpha} \quad \text{for all } \alpha \in \{0,1\}^n$$

**P4. Correlation:**

$$\sum_{X \in \{1,-1\}^n} f(X) \hat{f}(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha} \hat{a}_{\alpha}$$

**P5. Parseval's Property:**

$$\sum_{X \in \{1,-1\}^n} f^2(X) = 2^n \sum_{\alpha \in \{0,1\}^n} a_{\alpha}^2 = 2^n$$

# Polynomial Representation

## Basic Properties

$$f(X) = \sum_{\alpha \in \{0,1\}^n} a_{\alpha} X^{\alpha}$$

**P1. Orthogonality:**

$$\sum_{X \in \{1,-1\}^n} X^{\alpha} = \begin{cases} 2^n & \text{if } \alpha = \text{all-0 vector} \\ 0 & \text{else.} \end{cases}$$

**Proof:**

BY INDUCTION

# Polynomial Representation

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**Proof:**  $X^{\alpha}$  is a row in the matrix

$$H_4 = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}$$

$$H_{2^{n+1}} = \begin{bmatrix} H_{2^n} & H_{2^n} \\ H_{2^n} & -H_{2^n} \end{bmatrix}$$

The all-1 row balances because of the negation

# Polynomial Representation

## Basic Properties

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**Proof:**

Every row but the first is balanced (same number of 1's and -1's).

Also,

$X^{\alpha}$  is an XOR function of  $|\alpha|$  variables...

Q

**Important note:** all terms vanish but the constant term